THE GENERAL THEORY OF A CLASS OF LINEAR PARTIAL q-DIFFERENCE EQUATIONS*

BY

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INTRODUCTION

During the past fifteen years a great deal of work has been done in building up a general theory of the linear difference equation in one independent variable, or *linear ordinary difference equation.*† No study seems to have been made, however, of the linear difference equation in more than one independent variable, or *linear partial difference equation.*

Essentially two distinct types of ordinary difference equations have been subjected to investigation: (i) that usually spoken of simply as the *linear difference equation,*‡ and (ii) that which has received the name of *linear q-difference equation.*§ These suggest three types of linear partial difference equations in two independent variables, which (if we confine our attention to homogeneous equations) may be written as follows:

$$(A) \qquad \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}(x, y)\, g(x+m-\mu, y+n-\nu) = 0,$$

$$(B) \qquad \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}(x, y)\, g(q^{m-\mu} x, r^{n-\nu} y) = 0,$$

$$(C) \qquad \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}(x, y)\, g(x+m-\mu, r^{n-\nu} y) = 0,$$

in which q and r are constants, real or complex and different from zero; the $a_{\mu\nu}(x, y)$ are known analytic functions of x and y; and $g(x, y)$ is the

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† For a comprehensive survey of what has been done, see Nörlund, *Neuere Untersuchungen über Differenzengleichungen,* Encyklopädie der mathematischen Wissenschaften, vol. II_3, No. 6, Leipzig, Teubner, 1923, pp. 675–721.

‡ Cf. Birkhoff, *General theory of linear difference equations,* these Transactions, vol. 12 (1911), pp. 243–284.

§ Cf. Carmichael, *The general theory of linear q-difference equations,* American Journal of Mathematics, vol. 34 (1912), pp. 147–168.

function to be determined. To the equations (A), (B), and (C) we give respectively the names *linear partial pure difference equation, linear partial q-difference equation, and linear partial difference equation of the intermediate type.* If in (A), (B), and (C) μ be set equal to ν and m equal to n, the following important classes of these partial difference equations are obtained:

$$(A') \qquad \sum_{\nu=0}^{n} a_\nu(x, y)\, g(x+n-\nu, y+n-\nu) = 0,$$

$$(B') \qquad \sum_{\nu=0}^{n} a_\nu(x, y)\, g(q^{n-\nu} x, r^{n-\nu} y) = 0,$$

$$(C') \qquad \sum_{\nu=0}^{n} a_\nu(x, y)\, g(x+n-\nu, r^{n-\nu} y) = 0.$$

It is the purpose of the present paper to develop a general theory of equations of the class (B'); equations of classes (A') and (C') will be treated later. For the most part the methods used are along the lines of those employed by Birkhoff* in his work on ordinary difference equations. In § 1, formal series solutions are found; § 2 is devoted to proving the convergence of these series. In § 3 are considered certain cases to which the methods of §§ 1, 2 do not apply. The object of § 4 is to indicate that when $|q| = |r| = 1$, the equation in general has no analytic solution. The periodic functions which may be defined in terms of two solutions of an equation of this type are discussed in § 5. In § 6 a theorem is proved which is in part an inverse of the results obtained in the earlier work.

It should be observed that the chief results of the paper are immediately extensible to the case of an equation of the class (B') in N independent variables; i. e.,

$$\sum_{\nu=0}^{n} a_\nu(x, y, \ldots, w)\, g(q^{n-\nu} x, r^{n-\nu} y, \ldots, s^{n-\nu} w) = 0.$$

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1. The formal series solutions

The equation *of the nth order* of the class (B') will be defined as

$$(1) \qquad \sum_{\nu=0}^{n} p_\nu(x, y)\, g(q^{n-\nu} x, r^{n-\nu} y) = 0,$$

* Loc. cit.

in which q and r are real or complex constants not zero; the $p_\nu(x, y)$ are arbitrary known functions of the complex variables x and y and $p_0(x, y) \not\equiv 0$; and $g(x, y)$ is the function to be determined. It is important to observe first of all that

Every equation of the nth order of the class (B') can be solved by determining the solutions of an associated system of n equations of the first order of the class (B').

In fact if we define

$$g_i(x, y) = g(q^{i-1}x, r^{i-1}y) \qquad (i = 1, 2, \ldots, n),$$

the equation (1) is seen to be equivalent to the system

$$\begin{aligned} &g_i(qx, ry) = g_{i+1}(x, y) \qquad\qquad (i = 1, 2, \ldots, n-1), \\ (2)\qquad &g_n(qx, ry) = -\frac{p_n(x, y)}{p_0(x, y)} g_1(x, y) - \frac{p_{n-1}(x, y)}{p_0(x, y)} g_2(x, y) - \cdots \\ &\qquad\qquad \cdots - \frac{p_1(x, y)}{p_0(x, y)} g_n(x, y). \end{aligned}$$

We propose to show that an equation of the type (1) can in general be solved when the coefficients $p_\nu(x, y)$ are polynomials in x and y.* In that case the coefficients in the equivalent system (2), with which we shall deal, will be rational functions. It is unnecessary, however, for us to develop a theory for the solution of a system of equations with rational coefficients, for

The solution of a system of n equations of the first order of the class (B') with rational coefficients can in general be made to depend upon the solution of a second such system with polynomial coefficients.

Proof. Let

$$g_i(qx, ry) = \sum_{j=1}^{n} r_{ij}(x, y)\, g_j(x, y) \qquad (i = 1, 2, \ldots, n),$$

or in the matrix notation

$$G(qx, ry) = R(x, y)\, G(x, y), \tag{3}$$

be the given system. With no loss of generality it may and will be assumed that each rational function $r_{ij}(x, y)$ is expressed as the quotient of

* The problem is clearly the same as if the coefficients were taken to be rational functions.

relatively prime polynomials. If $d(x, y)$ is the least common denominator of the functions $r_{ij}(x, y)$, we have

$$R(x, y) = \frac{1}{d(x, y)} P(x, y),$$

where $P(x, y)$ is a matrix of polynomials. It will appear in the course of our later work (cf. §§ 2, 3) that, $d(x, y)$ being a polynomial, solutions can in general be found for the equation

$$g(qx, ry) = d(x, y)\, g(x, y).$$

Let one such solution be denoted by $\Phi(x, y)$ and set

$$G(x, y) = \frac{1}{\Phi(x, y)} \bar{G}(x, y);$$

then equation (3) becomes directly

$$\bar{G}(qx, ry) = P(x, y)\, \bar{G}(x, y).$$

It is the chief purpose of the present paper to find analytic solutions of the system

$$g_i(qx, ry) = \sum_{j=1}^{n} p_{ij}(x, y)\, g_j(x, y) \qquad (i = 1, 2, \ldots, n), \tag{4}$$

where $p_{ij}(x, y)$ is a polynomial

$$p_{ij}(x, y) = p_{ij} + p_{ij10}\, x + p_{ij01}\, y + \cdots + p_{ijlm}\, x^l y^m.$$

In the matrix notation the system (4) is written

$$G(qx, ry) = P(x, y)\, G(x, y). \tag{5}$$

Iteration of this equation suggests as symbolic solutions the two infinite products of matrices

$$P\left(\frac{x}{q}, \frac{y}{r}\right) P\left(\frac{x}{q^2}, \frac{y}{r^2}\right) P\left(\frac{x}{q^3}, \frac{y}{r^3}\right) \cdots,$$

$$P^{-1}(x, y)\, P^{-1}(qx, ry)\, P^{-1}(q^2x, r^2y) \cdots.$$

Certain modifications of these will play an important part in our work.

Among all the polynomials $p_{ij}(x, y)$ there will be one (or more) whose degree in x is the greatest; let that degree be μ: similarly there will be one (or more) whose degree in y is the greatest; let that degree be ν. Then the following determinantal equations will be called *the characteristic equations*:

for $(0, 0)$, $$|p_{ij} - \delta_{ij}\sigma| = 0;$$

for (∞, ∞), $$|p_{ij\mu\nu} - \delta_{ij}\sigma'| = 0;$$

for $(0, \infty)$, $$|p_{ij0\nu} - \delta_{ij}\sigma''| = 0;$$

for $(\infty, 0)$, $$|p_{ij\mu 0} - \delta_{ij}\sigma'''| = 0 \qquad \left(\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}\right).$$

The roots of these equations will be designated respectively by σ_i, σ_i', σ_i'', and σ_i''' $(i = 1, 2, \ldots, n)$.

Our attention will at first be confined to the case in which the roots σ_i, σ_i', σ_i'', and σ_i''' are all different from zero.* We shall make the further restriction that

$$\frac{\sigma_i}{\sigma_j} \neq q^\beta r^\gamma, \qquad \frac{\sigma_i'}{\sigma_j'} \neq q^\beta r^\gamma,$$

$$\frac{\sigma_i''}{\sigma_j''} \neq q^\beta r^\gamma, \qquad \frac{\sigma_i'''}{\sigma_j'''} \neq q^\beta r^\gamma \qquad (i \neq j),$$

for any β and γ which are positive integers or zero (but not both zero). Under these conditions and if we have $|q|, |r| \neq 1$, and if the roots of each characteristic equation are distinct, it is found by direct substitution and reckoning of the coefficients that the system (4) is satisfied formally (*a*) by each of n sets of functions

$$(6) \qquad \begin{aligned} s_{1j}(x, y) &= x^{\rho_j}[s_{1j} + s_{1j10}x + s_{1j01}y + \cdots], \\ &\cdots\cdots\cdots\cdots\cdots \\ s_{nj}(x, y) &= x^{\rho_j}[s_{nj} + s_{nj10}x + s_{nj01}y + \cdots] \end{aligned}$$

$$(j = 1, 2, \ldots, n),$$

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* A method by which solutions can frequently (in fact "in general" for the case of a single equation) be obtained when this condition does not hold will be given in § 3.

where the ϱ_j are determined by the relations

$$q^{\rho_j} = \sigma_j \quad \text{or} \quad \varrho_j = \log_q \sigma_j = \frac{\log \sigma_j}{\log q};$$

(*b*) by each of n sets of functions

$$\begin{aligned} s'_{1j}(x, y) &= q^{\mu(t^2-t)/2}\, r^{\nu(\tau^2-\tau)/2}\, x^{\rho'_j}\left[s'_{1j} + \frac{s'_{1j10}}{x} + \frac{s'_{1j01}}{y} + \cdots\right], \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ s'_{nj}(x, y) &= q^{\mu(t^2-t)/2}\, r^{\nu(\tau^2-\tau)/2}\, x^{\rho'_j}\left[s'_{nj} + \frac{s'_{nj10}}{x} + \frac{s'_{nj01}}{y} + \cdots\right] \end{aligned} \tag{7}$$

$$(j = 1, 2, \ldots, n),$$

in which

$$t = \frac{\log x}{\log q}, \qquad \tau = \frac{\log y}{\log r},$$

and the ϱ'_j are determined as

$$\varrho'_j = \frac{\log \sigma'_j}{\log q};$$

(*c*) by each of n sets of functions

$$\begin{aligned} s''_{1j}(x, y) &= r^{\nu(\tau^2-\tau)/2}\, x^{\rho''_j}\left[s''_{1j} + s''_{1j10}\, x + \frac{s''_{1j01}}{y} + \cdots\right], \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ s''_{nj}(x, y) &= r^{\nu(\tau^2-\tau)/2}\, x^{\rho''_j}\left[s''_{nj} + s''_{nj10}\, x + \frac{s''_{nj01}}{y} + \cdots\right] \end{aligned} \tag{8}$$

$$(j = 1, 2, \ldots, n),$$

where

$$\varrho''_j = \frac{\log \sigma''_j}{\log q};$$

and (*d*) by each of n sets of functions

$$\begin{aligned} s'''_{1j}(x, y) &= q^{\mu(t^2-t)/2}\, x^{\rho'''_j}\left[s'''_{1j} + \frac{s'''_{1j10}}{x} + s'''_{1j01}\, y + \cdots\right], \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ s'''_{nj}(x, y) &= q^{\mu(t^2-t)/2}\, x^{\rho'''_j}\left[s'''_{nj} + \frac{s'''_{nj10}}{x} + s'''_{nj01}\, y + \cdots\right] \end{aligned} \tag{9}$$

$$(j = 1, 2, \ldots, n),$$

in which

$$\varrho_j''' = \frac{\log \sigma_j'''}{\log q}.$$

Furthermore the sets of functions (6), and similarly (7), (8), and (9), are linearly independent. Throughout our work those determinations of $\log q$ and $\log r$ will be taken in which the coefficient of $\sqrt{-1}$ is positive or zero and less than 2π.*

We shall not assume, however, that the roots of each characteristic equation are distinct, but *we shall make the hypothesis that the formal series solutions* (6), (7), (8), *and* (9) *do exist and that they are such that the following determinantal inequalities hold*:

$$(10) \qquad \begin{aligned} d &= |s_{ij}| \neq 0, & d' &= |s'_{ij}| \neq 0, \\ d'' &= |s''_{ij}| \neq 0, & d''' &= |s'''_{ij}| \neq 0. \end{aligned}$$

The n sets of formal solutions (6) are then linearly independent and constitute a matrix $S(x, y)$ which is a formal matrix solution of the equation (5). The same statement may be made for the solutions (7), (8), and (9); let the respective matrices be denoted by $S'(x, y)$, $S''(x, y)$, and $S'''(x, y)$. It will be demonstrated (a) that if $|q|$ and $|r|$ are both greater or both less than unity, the series in the matrix $S(x, y)$ $[S'(x, y)]$ converge uniformly and absolutely in the neighborhood of any place† (x, y) for which x and y respectively lie inside [outside] certain associated circles r_1, r_2 $[R_1, R_2]$ about the origins in their respective planes; and (b) that if one of the quantities $|q|$ and $|r|$ is greater and the other less than unity, the series in the matrix $S''(x, y)$ $[S'''(x, y)]$ converge uniformly and absolutely in the neighborhood of any place (x, y) for which x and y lie respectively inside [outside] and outside [inside] certain associated circles r_1, R_2 $[R_1, r_2]$ about the origins in their respective planes.

The most general analytic matrix solution of (5) is given by

$$H(x, y) = G(x, y)\, A(x, y),$$

* It is to be noted that the power of x occurring immediately before the bracketed series in each set of formal series solutions is not a uniquely determined factor; in place of x^{ρ_j} in (6), for example, we may equally well use y^{ρ_j} or $(xy)^{\rho_j}$ or, in fact, any function $f(x, y)$ such that $f(qx, ry) = cf(x, y)$, where c is a constant.

† We shall find it convenient to think of the complex variables x and y as represented by points in two distinct planes. In conformity with the usual convention a pair of values of x and y will be spoken of as the *place* (x, y).

where $G(x, y)$ is any particular analytic matrix solution, and $A(x, y)$ is any arbitrary matrix of analytic functions which possesses the multiplicative period system* (q, r) and whose determinant $|A(x, y)|$ is not identically zero.

2. Existence theorems

Of the four cases $|q| \gtrless 1$, $|r| \gtrless 1$ which are to be considered in this section, that in which both absolute values are greater than unity is wholly typical. We shall therefore confine ourselves in giving detailed discussion to

The Case of $|q|, |r| > 1$.

It has been observed that the matrix $S(x, y)$ is a formal matrix solution of equation (5); that is, that

$$S(qx, ry) = P(x, y)\, S(x, y).$$

Now the determinant $|S(x, y)|$ may be written

$$x^{\rho_1+\rho_2+\cdots+\rho_n}\, [d + d_{10}\, x + d_{01}\, y + \cdots].$$

The element in the ith row and jth column of the inverse matrix is the quotient of the cofactor of the element in the jth row and ith column of this determinant by the determinant itself, and is therefore given by a series

$$\bar{s}_{ij}(x, y) = x^{-\rho_i}\, [\bar{s}_{ij} + \bar{s}_{ij10}\, x + \bar{s}_{ij01}\, y + \cdots]. \tag{11}$$

Let $T(x, y)$ denote the matrix obtained by breaking off the elements of $S(x, y)$ so as to retain in the series only the terms of degree less than or equal to $k-1$, or more generally by replacing $s_{ij}(x, y)$ by $t_{ij}(x, y)$, where the series in the latter is convergent for (x, y) in the vicinity of the place $(0, 0)$ and has the same terms as the series in $s_{ij}(x, y)$ up to and including those of degree $k-1$. We may then define a matrix $Q(x, y)$ by the relation

$$T(qx, ry) = Q(x, y)\, T(x, y).$$

* The matrix $A(x, y)$ shall be said to possess the *multiplicative period system* (q, r) if $A(qx, ry) = A(x, y)$.

The matrix $Q(x,y)$ is a matrix of functions $q_{ij}(x,y)$ each of which has an expansion in powers of x and y that agrees precisely with $p_{ij}(x,y)$ up to and including terms of degree $k-1$. This follows upon comparing

$$Q(x,y) = T(qx,ry)\,T^{-1}(x,y) \text{ and } P(x,y) = S(qx,ry)\,S^{-1}(x,y);$$

for $T(qx,ry)$ is the same as $S(qx,ry)$ up to and including terms of degree $k-1$, and similarly for $T^{-1}(x,y)$ and $S^{-1}(x,y)$, because $\bar{t}_{ij}(x,y)$ of $T^{-1}(x,y)$ is given by (11) up to and including terms of degree $k-1$. We therefore have

$$P(x,y) = Q(x,y) + M(x,y),$$

where $M(x,y)$ is a matrix of power series in x and y whose lowest degree terms are of degree k or higher and which converge in the vicinity of the place (0, 0). Then $N(x,y)$, defined by

$$M(x,y) = Q(x,y)\,N(x,y),$$

is a matrix of power series of the same type as those in $M(x,y)$. Hence

$$P(x,y) = Q(x,y)\,[I + N(x,y)],$$

in which I is the unit matrix and $N(x,y)$ is a matrix of power series each of which contains no term of degree less than k and which converges in the vicinity of (0, 0).

THEOREM I, A. *Form the product of matrices*

$$P_m(x,y) = P\left(\frac{x}{q},\frac{y}{r}\right) P\left(\frac{x}{q^2},\frac{y}{r^2}\right) \cdots P\left(\frac{x}{q^m},\frac{y}{r^m}\right) T\left(\frac{x}{q^m},\frac{y}{r^m}\right).$$

Each element of $P_m(x,y)$ converges, for k sufficiently large, to a definite limit function $u_{ij}(x,y)$, independent of k, as m becomes infinite. This function is analytic throughout the entire finite x- and y-planes except perhaps at $x=0$, and in this region is identical with the corresponding element $s_{ij}(x,y)$ of $S(x,y)$.

Proof. We may write

$$P_m(x,y) = T(x,y)\,\bar{P}_m(x,y),$$

where

$$\bar{P}_m(x,y) = \left[T^{-1}(x,y)\, P\left(\frac{x}{q},\frac{y}{r}\right) T\left(\frac{x}{q},\frac{y}{r}\right)\right]$$

$$\cdot\left[T^{-1}\left(\frac{x}{q},\frac{y}{r}\right) P\left(\frac{x}{q^2},\frac{y}{r^2}\right) T\left(\frac{x}{q^2},\frac{y}{r^2}\right)\right]\cdots$$

$$\cdots\left[T^{-1}\left(\frac{x}{q^{m-1}},\frac{y}{r^{m-1}}\right) P\left(\frac{x}{q^m},\frac{y}{r^m}\right) T\left(\frac{x}{q^m},\frac{y}{r^m}\right)\right].$$

The elements of $T(x,y)$ are polynomials or series convergent in the vicinity of $(0,0)$ (multiplied by a power of x). In order to show that the elements of $P_m(x,y)$ converge, it will be sufficient to prove that those of $\bar{P}_m(x,y)$ do. We proceed to a proof of this fact.

The matrix $\bar{P}_m(x,y)$ is the product of matrices of the type

$$T^{-1}(qx,ry)\, P(x,y)\, T(x,y) = I + T^{-1}(x,y)\, N(x,y)\, T(x,y).$$

The second term is a matrix

$$(x^{\rho_j-\rho_i}\lambda_{ij}(x,y)) = (\theta_{ij}(x,y)) = \Theta(x,y),.$$

where the $\lambda_{ij}(x,y)$ stand for power series in x and y whose lowest degree terms are of degree k or higher and which converge in the neighborhood of $(0,0)$.

The matrix $\bar{P}_m(x,y)$ may now be written

$$\left[I+\Theta\left(\frac{x}{q},\frac{y}{r}\right)\right]\left[I+\Theta\left(\frac{x}{q^2},\frac{y}{r^2}\right)\right]\cdots\left[I+\Theta\left(\frac{x}{q^m},\frac{y}{r^m}\right)\right]$$

$$= I+\sum_{k_1=1}^{m}\Theta\left(\frac{x}{q^{k_1}},\frac{y}{r^{k_1}}\right)+\sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m}\Theta\left(\frac{x}{q^{k_1}},\frac{y}{r^{k_1}}\right)\Theta\left(\frac{x}{q^{k_2}},\frac{y}{r^{k_2}}\right)+\cdots.$$

The ith element in the jth column $(i,j=1,2,\ldots,n)$ is

$$\begin{aligned}\bar{p}_{m;ij}(x,y) &= \delta_{ij}+\sum_{k_1=1}^{m}\theta_{ij}\left(\frac{x}{q^{k_1}},\frac{y}{r^{k_1}}\right)\\ &\quad+\sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m}\left[\sum_{\tau=1}^{n}\theta_{i\tau}\left(\frac{x}{q^{k_1}},\frac{y}{r^{k_1}}\right)\theta_{\tau j}\left(\frac{x}{q^{k_2}},\frac{y}{r^{k_2}}\right)\right]+\cdots,\end{aligned}\tag{12}$$

δ_{ij} denoting the element in the ith row and jth column of the unit matrix.

Let it next be noted that the product of two (and hence of N) matrices of the type of $\Theta(x, y)$ which differ only in the series λ is another matrix of the same type. For, if the two factor matrices are $\Theta(x, y)$ and $\Theta'(x, y)$ and their product is $\Theta''(x, y)$, we have (denoting by $[\lambda]$ a series of the type λ)

$$\begin{aligned}\theta''_{ij} &= \theta_{i1}\,\theta'_{1j} + \theta_{i2}\,\theta'_{2j} + \cdots + \theta_{in}\,\theta'_{nj} \\ &= x^{\rho_1-\rho_i}\,x^{\rho_j-\rho_1}[\lambda][\lambda] + x^{\rho_2-\rho_i}\,x^{\rho_j-\rho_2}[\lambda][\lambda] + \cdots + x^{\rho_n-\rho_i}\,x^{\rho_j-\rho_n}[\lambda][\lambda] \\ &= x^{\rho_j-\rho_i}[\lambda].\end{aligned}$$

This remark makes it clear that we have

$$\begin{aligned}x^{\rho_i-\rho_j}\,\bar{p}_{m;\,ij} - \delta_{ij} &= \sum_{k_1=1}^{m} \lambda_{ij}\left(\frac{x}{q^{k_1}}, \frac{y}{r^{k_1}}\right)\frac{1}{q^{(\rho_j-\rho_i)k_1}} \\ &+ \sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m}\left[\sum_{\tau=1}^{n}\lambda_{i\tau}\left(\frac{x}{q^{k_1}}, \frac{y}{r^{k_1}}\right)\lambda_{\tau j}\left(\frac{x}{q^{k_2}}, \frac{y}{r^{k_2}}\right)\frac{1}{q^{(\rho_\tau-\rho_i)k_1+(\rho_j-\rho_\tau)k_2}}\right] + \cdots.\end{aligned} \tag{13}$$

If p is equal to the smaller of the two quantities $|q|$ and $|r|$ and if x and y are inside certain associated circles r_1, r_2 about the origins in their respective planes, then the typical element in the lth term of the right-hand member of (13) is less in absolute value than

$$\frac{M^l}{p^{(k_1+k_2+\cdots+k_l)k}\left|q^{(\rho_\tau-\rho_i)k_1+(\rho_\sigma-\rho_\tau)k_2+\cdots+(\rho_j-\rho_\theta)k_l}\right|},$$

because when (x, y) is in this region all the functions $\lambda_{ij}(x, y)$ are less than some constant M. Furthermore if s is a positive integer assigned arbitrarily, we can take k so large that the inequality

$$p^k\left|q^{\rho_\alpha-\rho_\beta}\right| > p^s$$

will hold for $\alpha, \beta = 1, 2, \ldots, n$. Let any positive integer s be assigned and a suitable k be found; it follows that the series (13) will then be less term by term in absolute value than the series

$$\sum_{k_1=1}^{m}\frac{M}{p^{k_1 s}} + \sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m}\frac{n M^2}{p^{(k_1+k_2)s}} + \cdots.$$

As m is allowed to increase without limit this series approaches the value

$$\frac{1}{n}\left\{\left[\left(1+\frac{nM}{p^s}\right)\left(1+\frac{nM}{p^{2s}}\right)\cdots\right]-1\right\}.$$

The ratio

$$\frac{1}{p^{ns}} \div \frac{1}{p^{(n-1)s}} = \frac{1}{p^s},$$

since p is >1 and s is a positive integer, is less than unity. The element in the ith row and jth column of $\bar{P}_m(x, y)$, after being multiplied by $x^{\rho_i-\rho_j}$, therefore converges uniformly and absolutely to a function analytic in the neighborhood of places (x, y) such that $|x|$ is $<r_1$ and $|y|<r_2$. Hence each element of $P_m(x, y)$ converges in a similar manner in the neighborhood of places (x, y) such that $0 \neq |x| < r_1$, $|y| < r_2$.

We may write

$$P_m(x, y) = P\left(\frac{x}{q}, \frac{y}{r}\right) P\left(\frac{x}{q^2}, \frac{y}{r^2}\right) \cdots P\left(\frac{x}{q^t}, \frac{y}{r^t}\right) P_{m-t}\left(\frac{x}{q^t}, \frac{y}{r^t}\right)$$

and apply the argument as above to $P_{m-t}(x/q^t, y/r^t)$. It follows, since $|q|$ and $|r|$ are >1, that if any place (x, y) be assigned where x and y are in the finite parts of their respective planes and $x \neq 0$, t can be chosen so large that (x/q^t) $(\neq 0)$ and y/r^t will lie within the respective circles r_1 and r_2. We conclude that the elements of $P_m(x, y)$ converge to functions $u_{ij}(x, y)$ analytic over the entire finite parts of the x- and y-planes except at $x = 0$.

The functions $u_{ij}(x, y)$ are independent of k (so long as k is sufficiently large), because if k' is any second value for k and $T'(x, y)$ the matrix corresponding to $T(x, y)$, we may write

$$P'_m(x, y) = T(x, y)\left[T^{-1}(x, y)\, P\left(\frac{x}{q}, \frac{y}{r}\right) T\left(\frac{x}{q}, \frac{y}{r}\right)\right]$$

$$\cdot\left[T^{-1}\left(\frac{x}{q}, \frac{y}{r}\right) P\left(\frac{x}{q^2}, \frac{y}{r^2}\right) T\left(\frac{x}{q^2}, \frac{y}{r^2}\right)\right]\cdots$$

$$\cdots\left[T^{-1}\left(\frac{x}{q^{m-1}}, \frac{y}{r^{m-1}}\right) P\left(\frac{x}{q^m}, \frac{y}{r^m}\right) T'\left(\frac{x}{q^m}, \frac{y}{r^m}\right)\right].$$

Then if the elements be expanded in a sum like (12) and m be allowed to become infinite, the resulting multiple series for $u_{ij}(x,y)$ will be term by term identical with that obtained above.

It remains for us to show that if x and y are anywhere in the finite parts of their respective planes and $x \neq 0$, then $u_{ij}(x,y) \equiv s_{ij}(x,y)$. We shall first show this to be the case if $x(\neq 0)$ and y are respectively within the associated circles r_1, r_2. Let (x,y) be any place in this region. For this pair of values the limit function $\bar{u}_{ij}(x,y)$ of the ith element in the jth column of $\bar{P}_m(x,y)$ differs from δ_{ij} by a quantity less in absolute value than

$$|x^{\rho_j-\rho_i}|\frac{1}{n}\left\{\left[\left(1+\frac{nM}{p^s}\right)\left(1+\frac{nM}{p^{2s}}\right)\cdots\right]-1\right\},$$

which is itself less than

$$(1/n)\,|x^{\rho_j-\rho_i}|\left(e^{\sum\limits_{\nu=1}^{\infty}\frac{nM}{p^{\nu s}}}-1\right).$$

But

$$\sum_{\nu=1}^{\infty}\frac{1}{p^{\nu s}}=\frac{1}{p^s-1},$$

since $1/p^s$ is <1. Hence we have

$$|\bar{u}_{ij}(x,y)-\delta_{ij}|<\frac{|x^{\rho_j-\rho_i}|}{n}\left(e^{\frac{nM}{p^s-1}}-1\right).$$

If s (and with it k) be allowed to increase without limit, it is clear that $\bar{u}_{ij}(x,y)$ approaches δ_{ij}. Therefore is

$$u_{ij}(x,y)\equiv s_{ij}(x,y)\qquad (i,j=1,2,\ldots,n),$$

for we have shown $u_{ij}(x,y)$ to be independent of k, and as k becomes infinite $t_{ij}(x,y)$ becomes $s_{ij}(x,y)$. Thus the function $u_{ij}(x,y)$ is represented by $s_{ij}(x,y)$ for $0 \neq |x| < r_1$ and $|y| < r_2$. But the function $u_{ij}(x,y)$ is analytic without singularities of any kind over the entire finite x- and y-planes except at $x=0$. Hence the power series in $s_{ij}(x,y)$ converges for all finite values of x and y and represents $x^{-\rho_j}u_{ij}(x,y)$ throughout this region; it follows that $u_{ij}(x,y)\equiv s_{ij}(x,y)$ in the region specified.

The proof of the theorem as stated is thus complete.

We may consider in a similar manner the matrix $S'(x, y)$. It has already been noted that this matrix is a second formal solution of the matrix equation (5); that is, that

$$S'(qx, ry) = P(x, y)\, S(x, y),$$

which, for our immediate purposes, may be written more conveniently as

$$S'(x, y) = P^{-1}(x, y)\, S'(qx, ry).$$

The determinant $|S'(x, y)|$ is of the form

$$q^{n\mu(t^2-t)/2}\, r^{n\nu(\tau^2-\tau)/2}\, x^{\rho_1'+\rho_2'+\cdots+\rho_n'}\left[d' + \frac{d'_{10}}{x} + \frac{d'_{01}}{y} + \cdots\right].$$

The element in the ith row and jth column of the inverse matrix is consequently seen to be

$$\bar{s}'_{ij}(x, y) = q^{-\mu(t^2-t)/2}\, r^{-\nu(\tau^2-\tau)/2}\, x^{-\rho_i'}\left[\bar{s}'_{ij} + \frac{\bar{s}'_{ij10}}{x} + \frac{\bar{s}'_{ij01}}{y} + \cdots\right].$$

If $T'(x, y)$ denotes the matrix obtained by replacing $s'_{ij}(x, y)$ in $S'(x, y)$ by $t'_{ij}(x, y)$ where the series in the latter is known to be convergent in the neighborhood of (∞, ∞) and has the same terms as the series in $s'_{ij}(x, y)$ up to and including those of degree $k-1$ in $1/x$ and $1/y$, then we may define a matrix $Q'^{-1}(x, y)$ by the relation

$$T'(x, y) = Q'^{-1}(x, y)\, T'(qx, ry).$$

By comparison of

$$Q'^{-1}(x,y) = T'(x,y)\, T'^{-1}(qx,ry) \text{ and } P^{-1}(x,y) = S'(x,y)\, S'^{-1}(qx,ry),$$

we see that $Q'^{-1}(x, y)$ is a matrix of functions $\bar{q}'_{ij}(x, y)$ of the form

$$x^{-\mu}\, y^{-\nu}\left[\bar{q}'_{ij} + \frac{\bar{q}'_{ij10}}{x} + \frac{\bar{q}'_{ij01}}{y} + \cdots\right],$$

in which the series has exactly the same terms of degree less than k in $1/x$ and $1/y$ as has $x^{\mu} y^{\nu} \bar{p}^{ij}(x, y)$, $\bar{p}^{ij}(x, y)$ standing for the element in the ith row and jth column of the matrix $P^{-1}(x, y)$. It follows that

$$P^{-1}(x, y) = Q'^{-1}(x, y)[I + N'(x, y)],$$

where the elements of the matrix $N'(x, y)$ are power series in $1/x$ and $1/y$ which contain no terms of degree lower than k in these variables and which are convergent in the vicinity of (∞, ∞).

We may now state

THEOREM I, B. *Form the product of matrices*

$$\Pi'_m(x, y) = P^{-1}(x, y)\, P^{-1}(qx, ry) \cdots P^{-1}(q^{m-1}x, r^{m-1}y)\, T'(q^m x, r^m y).$$

Each element of $\Pi'_m(x, y)$ converges, for k sufficiently large, to a definite limit function $v_{ij}(x, y)$, independent of k, as m becomes infinite. This function is analytic throughout the finite x-and y-planes with the following exceptions: $x = 0$, $y = 0$, and places which are poles of an element of one of the matrices $P^{-1}(x, y)$, $P^{-1}(qx, ry), \cdots$. For x and y outside certain associated circles about the origins in their respective planes and $x, y \neq \infty$, $v_{ij}(x, y)$ is identical with the corresponding element $s'_{ij}(x, y)$ of $S'(x, y)$.

The proof of this theorem runs entirely parallel to that of Theorem I, A; we shall omit the details and pass on to the statement of the corresponding pair of theorems which hold in

The Case of $|q|, |r| < 1$.

Here the two symbolic solutions of the equation (5), $P(x/q, y/r) \cdot P(x/q^2, y/r^2) \cdots$ and $P^{-1}(x, y)\, P^{-1}(qx, ry) \cdots$, interchange the rôles which they played in the discussion of the preceding case. The theorems follow.

THEOREM I, C. *Form the product of matrices*

$$\Pi_m(x, y) = P^{-1}(x, y)\, P^{-1}(qx, ry) \cdots P^{-1}(q^{m-1}x, r^{m-1}y)\, T(q^m x, r^m y).$$

Each element of $\Pi_m(x, y)$ converges, for k sufficiently large, to a definite limit function $u_{ij}(x, y)$, independent of k, as m becomes infinite. This function is analytic throughout the finite x- and y-planes except perhaps at $x = 0$, and except at places which are poles of an element of one of the matrices $P^{-1}(x, y)$, $P^{-1}(qx, ry), \cdots$. For x and y inside certain associated circles r_1, r_2 about the origins in their respective planes and

$x \neq 0$, $u_{ij}(x, y)$ is identical with the corresponding element $s_{ij}(x, y)$ of $S(x, y)$.

THEOREM I, D. *Form the product of matrices*

$$P'_m(x, y) = P\left(\frac{x}{q}, \frac{y}{r}\right) P\left(\frac{x}{q^2}, \frac{y}{r^2}\right) \cdots P\left(\frac{x}{q^m}, \frac{y}{r^m}\right) T'\left(\frac{x}{q^m}, \frac{y}{r^m}\right).$$

Each element of $P'_m(x, y)$ converges, for k sufficiently large, to a definite limit function $v_{ij}(x, y)$, independent of k, as m becomes infinite. This function is analytic throughout the finite x- and y-planes except at $x = 0$ and $y = 0$, and in this region is identical with the corresponding element $s'_{ij}(x, y)$ of $S'(x, y)$.

Of course the $u_{ij}(x, y)$ and $v_{ij}(x, y)$ of this case are entirely distinct from the solutions of the preceding case for which the same notation was used.

The Cases of $|q| > 1$, $|r| < 1$ *and* $|q| < 1$, $|r| > 1$.

It is in the discussion of these cases that the matrices $S''(x, y)$ and $S'''(x, y)$ are of importance. We define a matrix $T''(x, y)$ $[T'''(x, y)]$ by replacing the series there by power series in x and $1/y$ $[1/x$ and $y]$ which are known to converge in the vicinity of the place $(0, \infty)$ $[(\infty, 0)]$ and which agree precisely with the series in $S''(x, y)$ $[S'''(x, y)]$ up to and including terms of degree $k-1$ in x and $1/y$ $[1/x$ and $y]$. Proceeding in entirely the same way as before we can prove, in each of the two cases, two theorems analogous to those above. We shall not pause to state the theorems here; it will suffice to remark that in both cases the series in the matrix $S''(x, y)$ $[S'''(x, y)]$ converge for x and y respectively inside [outside] and outside [inside] certain associated circles r_1, R_2 $[R_1, r_2]$ about the origins in their respective planes, and that by means of the equation (5) itself, analytic or meromorphic solutions are defined throughout a more extended region.

In the existence theorems for each of the four cases, use is made of but two of the four formal matrix solutions; for example, if $|q|$ and $|r|$ are both greater than unity, we have used only $S(x, y)$ and $S'(x, y)$. It would be of interest to know just what is the significance of the two remaining formal solutions.

3. The linear equation of the first order with polynomial coefficient for the case in which a characteristic equation has the root zero

It will be recalled that in § 1 one of the restrictions which it was found necessary to make in order to obtain the formal solutions (6), (7),

(8), and (9) is that the roots of each characteristic equation be all different from zero. An important particular case is that in which the system (4) consists of a single equation (i. e., when $n = 1$):

$$(14) \qquad \begin{aligned} g(qx, ry) &= p(x, y)\, g(x, y), \\ p(x, y) &= p + p_{10}\, x + p_{01}\, y + \cdots + p_{\mu\nu} x^{\mu} y^{\nu}. \end{aligned}$$

In this case if $p = 0$, the one root of the characteristic equation for (0,0) vanishes and no formal series solutions of the type (6) can be found. Then if $|q|$ and $|r|$ are both $>$ or <1, one of the two solutions that are usually obtainable is lacking. We desire to show, however, that a solution can in general be found to take its place. We shall restrict our discusssion entirely to this case, although the method introduced is of rather general application. It can be used directly if $|q|$ and $|r|$ are both $>$ or <1 and a solution is missing because $p_{\mu\nu} = 0$, and with slight modification if $|q|$ is >1 and $|r|$ is <1 or $|q|$ is <1 and $|r|$ is >1 and a solution is missing because $p_{0\nu} = 0$ or $p_{\mu 0} = 0$. It may also be possible at times to use the method effectively when n is >1 and one (or more) of the roots of a characteristic equation vanishes.

We shall accordingly assume $|q|, |r|>1$ or $|q|, |r|<1$ and proceed to obtain a solution for the equation (14) when $p = 0$. If $p(x, y)$ can be factored into $x^{\varkappa} y^{\lambda}$ (where $\varkappa$ and λ are positive integers or zero, but not both zero) multiplied by a polynomial $p'(x, y)$ whose constant term is not zero, then a solution of (14) can be obtained by solving the two equations

$$g_1(qx, ry) = x^{\varkappa} y^{\lambda} g_1(x, y), \qquad g_2(qx, ry) = p'(x, y)\, g_2(x, y),$$

and multiplying the solutions together. The first of these equations is satisfied by

$$g_1(x, y) = q^{\varkappa(t^2-t)/2}\, r^{\lambda(\tau^2-\tau)/2}, \text{ where } t = \frac{\log x}{\log q}, \ \tau = \frac{\log y}{\log r}.$$

As for the second equation, that is solvable at once by the methods of §§ 1, 2.

A geometrical interpretation of what it means for $p(x,y)$ to be factorable into $x^{\varkappa} y^{\lambda}\, p'(x, y)$ is worthy of notice. The terms of the polynomial $p(x, y)$ are of the form $x^m y^n$; let the point corresponding to each term be plotted in the m, n plane. Clearly each such point will lie in the first quadrant or upon the positive m- or n-axis, and a necessary and sufficient

condition that $p(x, y)$ be so factorable is that there be one point of this set in the m, n plane such that if there be drawn through it parallels to the axes, every other point of the set will lie above or upon the parallel to the m-axis and to the right or upon the parallel to the n-axis.

When $p(x, y)$ cannot thus be factored, our method of solving equation (14) will be to effect upon the variables x and y a transformation which will leave the given equation invariant in form but will change $p(x, y)$ into a polynomial $\bar{p}(\bar{x}, \bar{y})$ which can be factored in the desired manner. We propose to show that

$$(15) \qquad \begin{aligned} x &= \bar{x}^{\alpha}\,\bar{y}^{\beta}, \\ y &= \bar{x}^{\gamma}\,\bar{y}^{\delta}, \end{aligned} \qquad J = \alpha\delta - \beta\gamma \neq 0,$$

where each of the exponents is a suitably determined positive integer or zero, is such a transformation. It is readily seen that under this transformation the equation (14) becomes

$$(16) \qquad \bar{g}(\bar{q}\,\bar{x}, \bar{r}\,\bar{y}) = \bar{p}(\bar{x}, \bar{y})\,\bar{g}(\bar{x}, \bar{y}),$$

in which

$$\bar{q} = \left(\frac{q^{\delta}}{r^{\beta}}\right)^{1/J}, \qquad \bar{r} = \left(\frac{r^{\alpha}}{q^{\gamma}}\right)^{1/J}.$$

Hence it is our purpose to prove the following

THEOREM. *The quantities* $\alpha, \beta, \gamma, \delta$ *in* (15), *each of which is a positive integer or zero, can in general be so chosen that*

(*a*) $\bar{p}(\bar{x}, \bar{y})$ *will be factorable into* $\bar{x}^{\bar{\varkappa}}\,\bar{y}^{\bar{\lambda}}\,\bar{p}'(\bar{x}, \bar{y})$, *where* $\bar{p}'(\bar{x}, \bar{y})$ *has a constant term different from zero, and*

(*b*) $|\bar{q}|$ *and* $|\bar{r}|$ *will be both greater or both less than unity.*

Proof. The effect of the transformation (15) on the polynomial $p(x, y)$ may best be observed in the m, n plane. By (15) the term $x^m y^n$ goes into the term

$$\bar{x}^{\bar{m}}\,\bar{y}^{\bar{n}} = \bar{x}^{\alpha m+\gamma n}\,\bar{y}^{\beta m+\delta n}.$$

Thus the transformation of the m, n plane which corresponds to (15) is

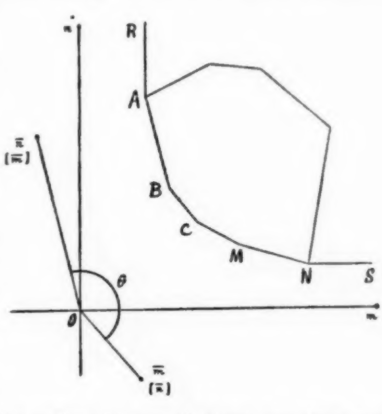
$$\begin{aligned} \bar{m} &= \alpha m + \gamma n, \\ \bar{n} &= \beta m + \delta n, \end{aligned} \qquad J = \alpha\delta - \beta\gamma \neq 0.$$

We shall think of this as a transformation of the axes, leaving the points of the plane fixed. Then the equations, referred to the m, n axes, of the $\bar{m}$, $\bar{n}$ axes are respectively

$$n = -\frac{\beta}{\delta} m, \qquad n = -\frac{\alpha}{\gamma} m.$$

These lines lie in the second and fourth quadrants, or at most, one of them may coincide with the m- or the n-axis. Furthermore, since for m and n positive $\bar{m}$ and $\bar{n}$ are also positive, it follows that the angle θ, less than π, between the positive $\bar{m}$- and $\bar{n}$-axes must include as a part of itself the angle between the positive m- and n-axes. By a proper choice of the exponents in (15) the $\bar{m}$- and $\bar{n}$-axes can be put into any desired position of this type so long as the slopes of the $\bar{m}$, $\bar{n}$ axes are rational (or in particular, infinite).

Let us now consider again our set of points in the m, n plane corresponding to terms of $p(x, y)$. A convex polygon can be drawn such that each vertex is a point of the set and every point of the set lies within or on the boundary of the polygon; let this polygon be constructed. Clearly it will have a vertex whose m-coördinate is less than that of any other vertex, or of any other but one. Call this point A; in case there are two such points, call the one with the larger n-coördinate A. Starting from this point proceed around the polygon in a counter-clockwise direction, lettering the successive vertices $B, C, \ldots$ until a vertex is reached whose n-coördinate is less than that of any other, or of any other except one. Let this last vertex be called N; or, if there are two such points, call the second one met (the one with the larger m-coördinate) N.



Now the slopes of all the lines AB, BC, ... are rational and negative, or in particular infinite or zero. If the slope of AB is not infinite, draw through A a parallel AR to the n-axis; if the slope of MN is not zero, draw through N a parallel NS to the m-axis. Then it is possible — given any vertex of the set $A, B, \ldots .$ N (exclusive of A if AB has an infinite slope and exclusive of N if MN has a zero slope), say B — to choose $\alpha, \beta, \gamma, \delta$ so that the $\bar{n}$- and $\bar{m}$-axes [the $\bar{m}$- and $\bar{n}$-axes] will

be parallel respectively to AR and AB, or to BA and BC, or ..., or to NM and NS. Any of these choices will yield a transformation (15) which will take $p(x, y)$ into a polynomial of the desired factorable type.

Secondly consider the condition (b). By hypothesis we have $|q|, |r| > 1$ or $|q|, |r| < 1$, and it is necessary that J be positive or negative. Let us assume $|q|, |r| > 1$ and try to determine $\alpha, \beta, \gamma, \delta$ so that J will be > 0 and condition (b), as well as condition (a), will be satisfied. It is readily seen that $|\bar{q}|$ and $|\bar{r}|$ will both be greater [less] than unity if and only if

$$\frac{\alpha}{\gamma} > \frac{\log|q|}{\log|r|} > \frac{\beta}{\delta} \qquad \left[\frac{\beta}{\delta} > \frac{\log|q|}{\log|r|} > \frac{\alpha}{\gamma}\right].$$

The second of these possibilities, however, necessitates J being < 0. Hence if $|q|$ and $|r|$ are > 1 and we are to use a transformation (15) in which J is > 0, a necessary and sufficient condition that the transformation satisfy the requirement (b) is that $\alpha, \beta, \gamma, \delta$ be determined to satisfy the inequality

$$\frac{\alpha}{\gamma} > \frac{\log|q|}{\log|r|} > \frac{\beta}{\delta}.$$

But $\log|q| \div \log|r|$ is a positive constant. Its negative therefore lies, in general, between the values of the slopes of two consecutive lines of the set $RA, AB, BC, \ldots, MN, NS$; in particular, its negative may be equal to the slope of one of the lines $AB, BC, \ldots, MN$ (AB excluded if its slope is infinite and MN if its slope is zero). In the general case, if $\alpha, \beta, \gamma, \delta$ be so chosen as to make the $\bar{n}$- and $\bar{m}$-axes parallel respectively to the two lines of the set $RA, \ldots, NS$ between the values of whose slopes $-\log|q| \div \log|r|$ lies, then the condition (b) as well as the condition (a) will be satisfied. In the particular case a solution cannot be obtained by this method.

Several observations should be made at this point. First, by determining $\alpha, \beta, \gamma, \delta$ so that the rôles of the $\bar{m}$ -and $\bar{n}$-axes in the preceding paragraph are interchanged, we obtain a transformation (15) satisfying (a) and (b) in which J is negative and $|\bar{q}|$ and $|\bar{r}|$ are both less than unity. Secondly, in the case of $|q|, |r| < 1$ a wholly similar discussion may be given. Thirdly, it is clear that in either case we can make $|\bar{q}|, |\bar{r}| > 1$, so that the words "or both less" may be stricken from part (b) in the statement of the theorem.

A solution for the equation (16) can be found by solving the two equations

$$\bar{g}_1(\bar{q}\bar{x}, \bar{r}\bar{y}) = \bar{x}^{\bar{\varkappa}}\bar{y}^{\bar{\lambda}}\bar{g}_1(\bar{x}, \bar{y}), \qquad \bar{g}_2(\bar{q}\bar{x}, \bar{r}\bar{y}) = \bar{p}'(\bar{x}, \bar{y})\bar{g}_2(\bar{x}, \bar{y})$$

and multiplying the solutions together; such a solution will be analytic throughout the finite $\bar{x}$- and $\bar{y}$-planes except at $\bar{x} = 0$ and $\bar{y} = 0$. Transforming this solution into terms of x and y by the inverse of transformation (15) we obtain a solution of (14) which is analytic throughout the finite x- and y-planes away from $x = 0$ and $y = 0$.

4. The case of $|q| = |r| = 1$

THEOREM. *When $|q| = |r| = 1$ the equation of the class (B') in general admits no analytic solution that is not identically zero.*

We distinguish two sub-cases: (i) when at least one of the constants q, r is not an nth root of unity for any positive integer n; (ii) when both q and r are roots of unity. The proofs will not be given here, for in each case a demonstration can be effected by the methods used by Carmichael* in proving the corresponding theorem for the linear ordinary q-difference equation; no very essential modification of Carmichael's work is needed beyond the introduction of a second independent variable.†

* Loc. cit., pp. 165–168.

† One point alone deserves special mention. In constructing a proof in subcase (i) it becomes necessary to show that, any positive integer M being given and any x and y being assigned, an integer n can be found which is $> M$ and such that the place $(q^n x, r^n y)$ is as near as you please to the place (x, y). This problem is equivalent to that of finding an $n > M$ such that we shall have

$$|q^n - 1| < \varepsilon \quad \text{and} \quad |r^n - 1| < \varepsilon,$$

$\varepsilon\ (>0)$ being preassigned and arbitrarily small. If we set $\alpha = \arg q$ and $\beta = \arg r$, this reduces to determining integers $n\ (> M)$, m, and p to satisfy at the same time the two inequalities

$$|n\alpha - m\cdot 2\pi| < \varepsilon' \quad \text{and} \quad |n\beta - p\cdot 2\pi| < \varepsilon'.$$

That such a determination is possible follows immediately from a generalized form of Theorem III, pp. 7–8, of Minkowski's *Diophantische Approximationen*, Leipzig, Teubner, 1907. If in Minkowski's proof (p. 7) we allow his variable z to take on the values $0, M, 2M, \ldots, t^2 M$, where M is an arbitrarily assigned positive integer, the rest of the proof goes through without change, and we obtain a theorem which may be stated as follows (for $M = 1$ it is Minkowski's): *Given two arbitrary real numbers a and b and any two positive integers M and t; there can always be found at least one integer n, $M \leqq n \leqq t^2 M$, and two other integers m and p such that the following inequalities hold simultaneously*:

$$|na - m| < \frac{1}{t}, \qquad |nb - p| < \frac{1}{t}.$$

5. The periodic functions

The discussion of this section will be confined to the case of $|q|, |r| > 1$; each of the other cases covered by the existence theorems of § 2 may be treated in the same manner.

It has been shown in § 2 that, under suitable restrictions, the matrix equation (5) admits of two matrix solutions, which may here be denoted for convenience by $S_{00}(x, y)$ and $S_{\infty\infty}(x, y)$ respectively. These solutions have been found to possess the following properties. $S_{00}(x, y)$ is analytic when x ($\neq 0$) is inside an arbitrarily large circle R_1 about the origin in the x-plane and y is inside an arbitrarily large circle R_2 about the origin in the y-plane, and is identically equal to $S(x, y)$. In a similar manner $S_{\infty\infty}(x, y)$ is analytic save for poles when x ($\neq \infty$) and y ($\neq \infty$) are respectively outside arbitrarily small circles r_1, r_2 about the origins in their respective planes. If (x, y) lies within the region of convergence of the formal series solutions (7), we have $S_{\infty\infty}(x, y) \equiv S'(x, y)$; if (x, y) lies outside that region, $S_{\infty\infty}(x, y)$ is given by

$$P^{-1}(x, y)\, P^{-1}(qx, ry) \cdots P^{-1}(q^{s-1}x, r^{s-1}y)\, S'(q^s x, r^s y),$$

where s is a suitable positive integer.

We now define a matrix $A(x, y)$ by the equation

$$S_{00}(x, y) = S_{\infty\infty}(x, y)\, A(x, y), \tag{17}$$

from which it is clear that $A(x, y)$ is a matrix of multiplicatively periodic functions such that

$$A(qx, ry) = A(x, y).$$

The elements of $A(x, y)$ are thus analytic, except perhaps for poles, in the region consisting of the x-plane between the circles r_1 and R_1 and of the y-plane between the circles r_2 and R_2; it will first be proved that no such poles can occur.

We have $A(x, y) = S_{\infty\infty}^{-1}(x, y) S_{00}(x, y)$. The elements of $S_{\infty\infty}^{-1}(x, y)$ may have singularities at the poles of elements of $S_{\infty\infty}(x, y)$ and at the zeros of the determinant $|S_{\infty\infty}(x, y)|$. But outside of certain definite associated circles $\bar{R}_1$, $\bar{R}_2$ the elements of $S_{\infty\infty}(x, y)$, so long as neither x nor y is actually infinite, have no poles, and outside certain other definite associated circles $\bar{\bar{R}}_1$, $\bar{\bar{R}}_2$ the determinant $|S_{\infty\infty}(x, y)|$, so long as neither

x nor y is actually infinite, does not vanish.* Now let any place (x_1, y_1) be chosen such that x_1 lies between r_1 and R_1 and y_1 lies between r_2 and R_2. We have

$$A(x_1, y_1) = A(qx_1, ry_1) = \cdots = A(q^s x_1, r^s y_1),$$

and s can be taken so large that $q^s x_1$ will lie outside the larger of the circles $\bar{R}_1$, $\bar{\bar{R}}_1$ and yet inside R_1, and $r^s y_1$ will lie outside the larger of the circles $\bar{R}_2$, $\bar{\bar{R}}_2$ but inside R_2. Hence $A(x, y)$ must be analytic at $(q^s x_1, r^s y_1)$ and therefore at (x_1, y_1). From this we conclude

THEOREM II, AB. *The elements of the matrix* $A(x, y)$ *defined by* (17) *are functions possessing the multiplicative period system* (q, r); *these functions are analytic in the region consisting of the x-plane between circles* r_1 *and* R_1 *about the origin and of the y-plane between circles* r_2 *and* R_2 *about the origin, where* r_1 *and* r_2 *are arbitrarily small and* R_1 *and* R_2 *are arbitrarily large.*

We shall find it worth while to examine these periodic functions a little more in detail. In order that they may be single-valued functions of position it will be convenient to think of each of the variables x and y as free to range over an infinitely many-sheeted Riemann surface of the type used for the logarithm, with branch points of infinite order at the origin and at infinity. Let us then make the transformation

$$x = q^t, \qquad y = r^\tau,$$

which maps the Riemann surfaces for x and y respectively in a one-to-one and conformal manner on the t-plane and τ-plane, and define

$$A(x, y) = B(t, \tau).$$

$B(t, \tau)$ is thus a matrix of functions single-valued and analytic over the region made up of the portions of the t-plane and τ-plane within arbitrarily large circles about the respective origins. Moreover the elements of $B(t, \tau)$ have additive, rather than multiplicative, period systems.

Let x now be made to traverse a positive circuit about the origin in its own plane (y remaining fixed). $S_{00}(x, y)$ becomes $S_{00}(x, y)K$, where K stands for the matrix $(e^{2\pi\rho_j\sqrt{-1}}\delta_{ij})$, δ_{ij} being as before the Kronecker δ.

* Cf. inequalities (10).

At the same time $S_{\infty\infty}(x, y)$ changes to

$$(-1)^{\mu} e^{2\pi\sqrt{-1}\,\mu t} e^{-2\pi^2\mu/\log q} S_{\infty\infty}(x, y) L,$$

where L is the matrix $(e^{2\pi\rho_j'\sqrt{-1}}\,\delta_{ij})$. Thus when x has made a positive circuit of the origin, $A(x, y)$ is replaced by

$$(-1)^{\mu} e^{-2\pi\sqrt{-1}\,\mu t} e^{2\pi^2\mu/\log q} L^{-1} A(x, y) K.$$

If, then, $a_{ij}(x, y)$ is the element in the ith row and jth column of $A(x, y)$, the passage of x around such a circuit changes $a_{ij}(x, y)$ to

$$(-1)^{\mu} e^{-2\pi\sqrt{-1}\,\mu t} e^{2\pi^2\mu/\log q} e^{2\pi\sqrt{-1}(-\rho_i'+\rho_j)} a_{ij}(x, y).$$

Similarly if y be caused to traverse a circuit about the origin in the positive sense (x remaining fixed), $a_{ij}(x, y)$ is replaced by

$$(-1)^{\nu} e^{-2\pi\sqrt{-1}\,\nu\tau} e^{2\pi^2\nu/\log r} a_{ij}(x, y).$$

It follows that if $b_{ij}(t, \tau)$ denotes the element in the ith row and jth column of $B(t, \tau)$, then

$$b_{ij}(t+1, \tau+1) = b_{ij}(t, \tau),$$

$$(18) \quad b_{ij}\left(t + \frac{2\pi\sqrt{-1}}{\log q}, \tau\right) = (-1)^{\mu} e^{-2\pi\sqrt{-1}(\mu t+\rho_i'-\rho_j)+2\pi^2\mu/\log q} b_{ij}(t, \tau),$$

$$b_{ij}\left(t, \tau + \frac{2\pi\sqrt{-1}}{\log r}\right) = (-1)^{\nu} e^{-2\pi\sqrt{-1}\,\nu\tau+2\pi^2\nu/\log r} b_{ij}(t, \tau).$$

Thus the elements of the matrix $B(t, \tau)$ are seen to be triply periodic, or perhaps one might better say "triply quasi-periodic." As such these functions are a special case of the functions treated by Cousin* in a memoir in which he discusses, with only slight restrictions, the properties of the general meromorphic function of two complex variables whose zeros admit

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* *Sur les fonctions triplement périodiques de deux variables,* Acta Mathematica, vol. 33 (1910), pp. 105–232.

three systems of periods.* These restrictions are in general satisfied by the functions $b_{ij}(t,\tau)$. The functions $b_{ij}(t,\tau)$ are a more restricted type than those dealt with by Cousin in that they are entire, rather than meromorphic functions and in that the exponents of e in the equations (18) are linear functions of t and τ rather than the wholly general entire functions which appear in the corresponding equations of Cousin. The Cousin paper also shows how a triply periodic function can be expressed in explicit form in terms of series of the type

$$\sum_{n=-\infty}^{+\infty} \Phi_n(t)\, e^{n\tau}.$$

in which $\Phi_n(t)$ is essentially an exponential function of t multiplied by a product of θ-functions of $t - a_i^{(n)}$ ($a_i^{(n)}$ being a constant ($i = 1, 2, \ldots, k$)). Hence the functions $b_{ij}(t,\tau)$ are expressible in such a manner as this, which is remotely analogous to the explicit form found by Birkhoff† for the periodic functions that arise in the theory of the linear ordinary q-difference equation.

6. A more general problem; an inverse theorem

In the four matrices of formal series solutions as they were found in § 1, there seems to be a lack of symmetry. This lack, however, is only apparent, for in defining our characteristic equations we might equally well have proceeded as follows. Among all the polynomials $p_{ij}(x,y)$ there will be one (or more) which contains $x\,[y]$ to a maximum power; let that power be $\mu\,[\nu]$; similarly there will be one (or more) which contains $x\,[y]$ to a minimum power; let that power be $\varkappa\,[\lambda]$. Then the characteristic equations are

for $(0,0)$, $$|p_{ij\varkappa\lambda} - \delta_{ij}\,\sigma\,| = 0;$$

for (∞,∞), $$|p_{ij\mu\nu} - \delta_{ij}\,\sigma'\,| = 0;$$

for $(0,\infty)$, $$|p_{ij\varkappa\nu} - \delta_{ij}\,\sigma''\,| = 0;$$

for $(\infty,0)$, $$|p_{ij\mu\lambda} - \delta_{ij}\,\sigma'''| = 0;$$

---

* The zeros of $f(t,\tau)$ are said to admit the system of periods (a,b) if $f(t+a,\tau+b) = e^{g(t,\tau)} f(t,\tau)$, where $g(t,\tau)$ is an entire function.

† *The generalized Riemann problem for* ..., Proceedings of the American Academy of Arts and Sciences, vol. 49 (1913), pp. 561-564.

and the formal matrix solutions are found to be

$$S(x,y) = (q^{\varkappa(t^2-t)/2}\, r^{\lambda(\tau^2-\tau)/2}\, x^{\rho_j}\ [s_{ij} + s_{ij10}\,x + s_{ij01}\,y + \cdots]),$$

$$S'(x,y) = \left(q^{\mu(t^2-t)/2}\, r^{\nu(\tau^2-\tau)/2}\, x^{\rho_j'}\ \left[s_{ij}' + \frac{s_{ij10}'}{x} + \frac{s_{ij01}'}{y} + \cdots\right]\right),$$

$$S''(x,y) = \left(q^{\varkappa(t^2-t)/2}\, r^{\nu(\tau^2-\tau)/2}\, x^{\rho_j''}\ \left[s_{ij}'' + s_{ij10}''\,x + \frac{s_{ij01}''}{y} + \cdots\right]\right),$$

$$S'''(x,y) = \left(q^{\mu(t^2-t)/2}\, r^{\lambda(\tau^2-\tau)/2}\, x^{\rho_j'''}\left[s_{ij}''' + \frac{s_{ij10}'''}{x} + s_{ij01}'''\,y + \cdots\right]\right).$$

In this form there is evidently complete symmetry, and we note that when $\varkappa = \lambda = 0$ the matrices reduce to those of § 1.

It immediately becomes clear, after the preceding remark, that the polynomials $p_{ij}(x,y)$ may just as well be taken as polynomials in x, $1/x$, y, and $1/y$. In that event the formal work of the foregoing paragraph is unchanged, but $\varkappa$ and λ are negative (i. e., if terms in $1/x$ and $1/y$ actually occur). Furthermore it should be observed that in the cases of $|q|,\ |r| > 1$ and of $|q|,\ |r| < 1$, existence theorems can be proved almost entirely as above if the elements of the matrix are any functions whatever which are expressible in each of the following forms, the first valid in the vicinity of $(0,0)$ and the second valid in the vicinity of (∞,∞):

$$\text{(19)} \qquad \begin{aligned} &x^{\varkappa}\, y^{\lambda}\ [\text{convergent power series in } x \text{ and } y],\\ &x^{\mu}\, y^{\nu}\left[\text{convergent power series in } \frac{1}{x} \text{ and } \frac{1}{y}\right] \end{aligned}$$

(provided, of course, the restrictive conditions imposed in § 1 are satisfied); the only difference in the results is that the solutions obtained may be valid only in definitely limited regions about $(0,0)$ and (∞,∞) respectively. It is clear that a similar remark applies in the two remaining cases, $|q| > 1$, $|r| < 1$ and $|q| < 1$, $|r| > 1$.

We are now in a position to prove a theorem which is in part an inverse of some of the results that have been found in this and preceding sections.

THEOREM III, ABCD. *Let $G(x,y)$ and $H(x,y)$ be two matrices whose elements, after multiplication by $q^{-\varkappa(t^2-t)/2}\, r^{-\lambda(\tau^2-\tau)/2}\, x^{-\rho_j}$ in the case of $G(x,y)$ and by $q^{-\mu(t^2-t)/2}\, r^{-\nu(\tau^2-\tau)/2}\, x^{-\rho'_j}$ in the case of $H(x,y)$, are single-valued functions analytic save for poles over the extended x- and y-planes with the exception of the places $(0,0)$ and (∞,∞) and such that*

$$\lim_{\substack{x=0\\y=0}} g_{ij}(x,y)\, q^{-\varkappa(t^2-t)/2}\, r^{-\lambda(\tau^2-\tau)/2}\, x^{-\rho_j} = s_{ij}, \qquad \begin{pmatrix}|q|,\ |r|>1 \textit{ or}\\ |q|,\ |r|<1\end{pmatrix}$$

$$\lim_{\substack{x=\infty\\y=\infty}} h_{ij}(x,y)\, q^{-\mu(t^2-t)/2}\, r^{-\nu(\tau^2-\tau)/2}\, x^{-\rho'_j} = s'_{ij},$$

where the s's and ϱ's are constants subject to the conditions

$$|s_{ij}| \neq 0, \qquad |s'_{ij}| \neq 0;$$

$$\varrho_i - \varrho_j \neq \beta + \gamma\frac{\log r}{\log q}, \qquad \varrho'_i - \varrho'_j \neq \beta + \gamma\frac{\log r}{\log q}$$

for any β and γ that are positive or negative integers or zero $(i \neq j)$; and where $\varkappa$, λ, μ and ν are integers (or zero) and

$$t = \frac{\log x}{\log q}, \qquad \tau = \frac{\log y}{\log r}.$$

Also let the two matrices be connected by the relation

$$G(x,y) = H(x,y)\, A(x,y),$$

in which $A(x,y)$ is a matrix of functions possessing the multiplicative period system (q,r). Then $G(x,y)$ and $H(x,y)$ are solutions of a system (5) *in which the elements of $P(x,y)$ are rational functions.*

Proof. By hypothesis we have

$$H^{-1}(qx,ry)\, G(qx,ry) = H^{-1}(x,y)\, G(x,y).$$

Let $P(x,y)$ denote the product of matrices

$$G(qx,ry)\, G^{-1}(x,y) = H(qx,ry)\, H^{-1}(x,y);$$

its elements are single-valued and analytic except for poles over the extended x- and y-planes apart from the places $(0, 0)$ and (∞, ∞).

Upon calculating the elements $\bar{g}_{ij}(x, y)$ $[\bar{h}_{ij}(x, y)]$ of $G^{-1}(x, y)$ $[H^{-1}(x, y)]$, it becomes clear, by virtue of our hypotheses, that

$$\lim_{\substack{x=0\\y=0}} \bar{g}_{ij}(x, y)\, q^{\kappa(t^2-t)/2}\, r^{\lambda(\tau^2-\tau)/2}\, x^{\rho_i} = \bar{s}_{ij}$$

$$\left[\lim_{\substack{x=\infty\\y=\infty}} \bar{h}_{ij}(x, y)\, q^{\mu(t^2-t)/2}\, r^{\nu(\tau^2-\tau)/2}\, x^{\rho'_i} = \bar{s}'_{ij}\right],$$

where $\bar{s}_{ij}$ $[\bar{s}'_{ij}]$ is the element in the ith row and jth column of the inverse matrix of (s_{ij}) $[(s'_{ij})]$. Hence, if (x, y) is in the neighborhood of $(0, 0)$, we have

$$p_{ij}(x, y) = \sum_{\alpha=1}^{n} g_{i\alpha}(qx, ry)\, \bar{g}_{\alpha j}(x, y) = x^{\kappa} y^{\lambda} \sum_{\alpha=1}^{n} q^{\rho_\alpha}(s_{i\alpha}\bar{s}_{\alpha j} + \eta_{ij\alpha}),$$

where $\eta_{ij\alpha}(i, j, \alpha = 1, 2, \ldots, n)$ approaches zero as (x, y) approaches $(0, 0)$. Similarly, if (x, y) is in the neighborhood of (∞, ∞), we have

$$p_{ij}(x, y) = x^{\mu} y^{\nu} \sum_{\alpha=1}^{n} q^{\rho'_\alpha}(s'_{i\alpha}\bar{s}'_{\alpha j} + \eta'_{ij\alpha}),$$

in which $\eta'_{ij\alpha}(i, j, \alpha = 1, 2, \ldots, n)$ approaches zero as (x, y) approaches (∞, ∞). It follows that the elements of $P(x, y)$ are analytic or have poles at $(0, 0)$ and at (∞, ∞). Consequently they are analytic except for poles over the extended x- and y-planes without exception, and are rational functions of x and y.

We have, then, the two equations

$$G(qx, ry) = P(x, y)\, G(x, y), \tag{20}$$

$$H(qx, ry) = P(x, y)\, H(x, y). \tag{21}$$

From the preceding paragraph it follows that the elements of $P(x, y)$ have at $(0, 0)$ and at (∞, ∞) the requisite forms (19). It remains to show that the roots of the characteristic equations for $(0, 0)$ and for (∞, ∞)

satisfy the conditions imposed in § 1. To do this let us rewrite (20) in the form

$$g_{ij}(qx, ry) = \sum_{u=1}^{n} p_{iu}(x, y)\, g_{uj}(x, y) \qquad (i, j = 1, 2, \ldots, n).$$

For any particular i this system provides n linear non-homogeneous equations for the determination of the n functions $p_{iu}(x, y)$ $(u = 1, 2, \ldots, n)$. If they be solved for $p_{iu}(x, y)$ and (x, y) be allowed to approach $(0, 0)$, the coefficient in the leading term of the expansion for $p_{iu}(x, y)$ at $(0, 0)$ being denoted by $p_{iu\varkappa\lambda}$, we have

$$p_{iu\varkappa\lambda} = \frac{\left| s_{lm}^{(iu)} \right|}{\left| s_{lm} \right|},$$

where $\left| s_{lm}^{(iu)} \right|$ stands for the determinant obtained from $|s_{lm}|$ by replacing the elements of the uth row by $q^{\rho_j} s_{ij} (j = 1, 2, \ldots, n)$. (It is clearly possible thus to solve, because by hypothesis $|s_{lm}| \neq 0$.) The characteristic equation for $(0, 0)$ is

$$| p_{iu\varkappa\lambda} - \delta_{iu}\sigma | = 0;$$

upon examination its roots are seen to be $q^{\rho_1}, q^{\rho_2}, \ldots, q^{\rho_n}$. In the same manner we may show from (21) that, if $p_{iu\mu\nu}$ is the coefficient in the leading term of the expansion for $p_{iu}(x, y)$ at (∞, ∞), then we have

$$p_{iu\mu\nu} = \frac{\left| s_{lm}^{\prime (iu)} \right|}{\left| s_{lm}^{\prime} \right|},$$

and that the roots of the characteristic equation for (∞, ∞),

$$| p_{iu\mu\nu} - \delta_{iu}\sigma' | = 0,$$

are $q^{\rho'_1}, q^{\rho'_2}, \ldots, q^{\rho'_n}$. It then follows from our hypotheses that the matrix satisfies all the conditions that it has been found necessary, in our earlier work, to impose upon the matrix of coefficients in the system (5). This completes the proof of the theorem.

If one of the two numbers $|q|$ and $|r|$ is greater and the other less than unity, an entirely similar theorem can be stated and proved.

In conclusion, we may call attention to the fact that a discussion of the equation (B') such as has been given covers also the cases of equations (A') and (C') for certain types of coefficient functions. In the first place, subjecting equation (5) to the transformation $x = q^t, y = r^\tau$, we see at once that our discussion disposes of the case of a system of equations of the first order of the class (A') when the coefficients are polynomials (or such other functions as have been noted above) in q^x and r^y. Secondly, if in (5) we set $x = q^t$, $y = \tau$ $[x = t,\ y = r^\tau]$, we find that our work has covered the case of a system of equations of the first order of the class (C') with coefficients which are polynomials (or such other functions as we have noted above) in q^x and y $[x$ and $r^y]$.

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THE SUMMABILITY OF THE TRIPLE FOURIER SERIES AT POINTS OF DISCONTINUITY OF THE FUNCTION DEVELOPED*

BY

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In a previous paper† the author has studied the summability of the development of a function of three variables in a triple Fourier series at points of continuity of the function developed, using the method originated by Fejér and applied by him to problems involving simple series. For the latter series the proof of the summability at points of discontinuity of the first kind (finite jumps) is quite similar to that for points of continuity, and the two cases can be treated by means of a single discussion. In passing to the case of the double Fourier series, it is found that the study of the behavior of the series at points of discontinuity of an analogous type presents difficulties and complications that do not arise in connection with points of continuity.‡ When we go on to the case of triple series, we find that the difficulties and complications of the corresponding problem are still further increased.

In the present paper a study is made of the summability of the triple Fourier series at points of discontinuity of the type that would be apt to arise in physical applications. This includes discontinuities lying on plane or curved surfaces, and such that the function to be developed approaches the same value as we approach the point along any path lying entirely within the region of continuity.

The definition of summability used in the discussion of the triple series is analogous to that of Cesàro for the simple series. Designating by s_{lmn} the sum of the lmn terms of the triple series $\sum a_{lmn}$ lying in a rectangular parallelepiped l terms high, m terms broad, and n terms deep, in the upper, left-hand, forward corner of this series, and forming

$$(1)\quad S_{lmn}^{(r)} = \sum_{i=1,j=1,k=1}^{l,m,n} \frac{\Gamma(r+l-i)}{\Gamma(r)\Gamma(l-i+1)} \cdot \frac{\Gamma(r+m-j)}{\Gamma(r)\Gamma(m-j+1)} \cdot \frac{\Gamma(r+n-k)}{\Gamma(r)\Gamma(n-k+1)} s_{ijk},$$

$$(2)\qquad A_{lmn}^{(r)} = \frac{\Gamma(l+r)}{\Gamma(r+1)\Gamma(l)} \cdot \frac{\Gamma(m+r)}{\Gamma(r+1)\Gamma(m)} \cdot \frac{\Gamma(n+r)}{\Gamma(r+1)\Gamma(n)},$$

* Presented to the Society, December 28, 1923.

† Annals of Mathematics, ser. 2, vol. 24 (1922), pp. 141–166.

‡ A discussion of this case has been given by Professor C. N. Moore (cf. Mathematische Annalen, vol. 74 (1913), pp. 557–572.

we say that the triple series is summable (Cr) to the limit

$$\lim_{l,m,n\to\infty} \frac{S_{lmn}^{(r)}}{lmn} \tag{3}$$

and has the value of that limit, providing such a limit exists.

I. Discontinuities along plane surfaces

The triple Fourier series corresponding to the function $f(x, y, z)$ may by written in the form

$$\sum_{l=1,m=1,n=1}^{\infty,\infty,\infty} \frac{1}{2^{E(1/l)+E(1/m)+E(1/n)}\pi^3} \tag{4}$$

$$\cdot \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x', y', z')\, P_{lmn}(x, y, z, x', y', z')\, dx'\, dy'\, dz',$$

where

$$P_{lmn}(x, y, z, x', y', z')$$
$$= \cos[(l-1)(x'-x)] \cos[(m-1)(y'-y)] \cos[(n-1)(z'-z)]$$

and $E(u)$ represents the largest integer contained in u.

We wish to investigate first the behavior of the series at a point of discontinuity such that all other points of discontinuity in its neighborhood lie on a plane passing through that point and such that $f(x, y, z)$ approaches a definite value as we approach the point of discontinuity from either side of the plane. Several lemmas are necessary before we can prove the fundamental theorem. The first two are merely stated, their proofs appearing in a previous paper by the writer.*

Lemma 1. *Let R be a region in space lying within the cube whose sides are $\alpha = \pm(\pi-\varrho_1)$, $\beta = \pm(\pi-\varrho_1)$, $\gamma = \pm(\pi-\varrho_1)$, and outside the sphere of radius ϱ_2 whose center is at the origin, where ϱ_1 and ϱ_2 are two positive constants whose sum is less than π. Then if $\varphi(\alpha, \beta, \gamma)$ is finite and integrable*† *in R, the limit*

$$\lim_{l,m,n\to\infty} \frac{1}{lmn}\int\int\int_R \varphi(\alpha, \beta, \gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\, d\alpha\, d\beta\, d\gamma$$

will exist and be equal to zero.

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* Loc. cit., pp. 154-156, 157-158.

† Here and elsewhere thoughout the paper, when it is said that a function is integrable, it is meant that it has an integral according to the definition of Lebesgue.

LEMMA 2. *Let R be a region within the cube described in Lemma 1, and such that the point $\alpha = \beta = \gamma = 0$ lies within or on the boundary of R. Then if $\varphi(\alpha, \beta, \gamma)$ is finite and integrable in R, and if*

$$\lim_{\alpha, \beta, \gamma \to 0} \varphi(\alpha, \beta, \gamma) = 0,$$

the limit

$$\lim_{l, m, n \to \infty} \frac{1}{lmn} \int\int\int_R \varphi(\alpha, \beta, \gamma) \frac{\sin^2 l\alpha}{\sin^2 \alpha} \frac{\sin^2 m\beta}{\sin^2 \beta} \frac{\sin^2 n\gamma}{\sin^2 \gamma} \, d\alpha \, d\beta \, d\gamma$$

will exist and be equal to zero.

LEMMA 3. *If g, h and k are positive numbers less than π, the limit*

$$\lim_{l, m, n \to \infty} \frac{1}{lmn} \int_0^g \int_0^h \int_0^k \frac{\sin^2 l\alpha}{\operatorname{lin}^2 \alpha} \frac{\sin^2 m\beta}{\sin^2 \beta} \frac{\sin^2 n\gamma}{\sin^2 \gamma} \, d\alpha \, d\beta \, d\gamma \tag{5}$$

and the seven others in which the limits of integration are respectively $-g$ to 0, 0 to h, 0 to k; 0 to g, $-h$ to 0, 0 to k; 0 to g, 0 to h, $-k$ to 0; $-g$ to 0, $-h$ to 0, 0 to k; $-g$ to 0, 0 to h, $-k$ to 0; 0 to g, $-h$ to 0, $-k$ to 0; $-g$ to 0, $-h$ to 0, $-k$ to 0, will each exist and be equal to $\frac{1}{8}$.

The proof for the first limit follows easily from Lemma 1 and a result due to Fejér.* The succeeding seven cases may be reduced to the first by a suitable change of variable. An analogous proof is given under Lemma 5 of the writer's previous paper.

LEMMA 4. *If g, h and k are positive numbers less than π and we divide the parallelepiped whose sides are $\alpha = \pm g$, $\beta = \pm h$, $\gamma = \pm k$ into two parts R_1 and R_2 by passing a plane through the diagonal joining two opposite vertices, the limits*

$$\begin{aligned} &\lim_{l, m, n \to \infty} \left\{ \frac{1}{lmn\pi^3} \int\int\int_{R_1} \frac{\sin^2 l\alpha}{\sin^2 \alpha} \frac{\sin^2 m\beta}{\sin^2 \beta} \frac{\sin^2 n\gamma}{\sin^2 \gamma} \, d\alpha \, d\beta \, d\gamma \right\}, \\ &\lim_{l, m, n \to \infty} \left\{ \frac{1}{lmn\pi^3} \int\int\int_{R_2} \frac{\sin^2 l\alpha}{\sin^2 \alpha} \frac{\sin^2 m\beta}{\sin^2 \beta} \frac{\sin^2 n\gamma}{\sin^2 \gamma} \, d\alpha \, d\beta \, d\gamma \right\} \end{aligned} \tag{6}$$

will exist and will each equal $\frac{1}{2}$.

Suppose the plane passes through the diagonal joining the vertices (g, h, k) and $(-g, -h, -k)$. In general this plane will cut through two pairs of faces. Suppose for the sake of definiteness that we assume it cuts the

* Mathematische Annalen, 58 (1904), p. 55.

faces $\alpha = \pm g, \beta = \pm h$. We shall chose R_1 as the region included between the dividing plane and the planes $\alpha = \pm g$, $\beta = \pm h$, $\gamma = -k$, and show that the first limit in (6) exists and is equal to $\frac{1}{2}$. It will then follow that the second of these limits exists and is equal to $\frac{1}{2}$, as we know that the sum of the two expressions in brackets approaches the limit unity as l, m and n become infinite.

The equation of the plane may be written in the form $\gamma = p\alpha + q\beta$ where p and q are constants. The quantity in brackets in the first of expressions (6) may then be written

$$
\begin{aligned}
(7)\qquad \frac{1}{lmn\pi^3}\Big\{\int_0^g\int_0^h\int_{-k}^0 + \int_0^g\int_0^h\int_0^{p\alpha+q\beta} + \int_0^g\int_{-h}^0\int_{-k}^{p\alpha+q\beta} \\
+ \int_{-g}^0\int_{-h}^0\int_{-k}^{p\alpha+q\beta} + \int_{-g}^0\int_0^h\int_{-k}^{p\alpha+q\beta}\Big\},
\end{aligned}
$$

where the quantity under the integral sign in each case is the same as that under the integral sign in the expression (6).

The first term of (7) approaches $\frac{1}{8}$ as a limit as l, m and n become infinite, by Lemma 3. If we make the substitution $\alpha = -\alpha'$, $\beta = -\beta'$, $\gamma = -\gamma'$ in the third and fourth terms, the third combines with the fifth to form an expression which by Lemma 3 approaches $\frac{1}{4}$ as a limit as l, m and n become infinite, and the fourth combines with the second to form an expression which approaches $\frac{1}{8}$ as l, m and n become infinite, by the same lemma. Hence in this case the limit of the first of expressions (6) exists and is equal to $\frac{1}{2}$.

The plane through the diagonal joining the vertices (g, h, k) and $(-g, -h, -k)$ may also be perpendicular to one of the planes, say to $\gamma = k$. In this case we take R_1 to be the region included between the plane through the diagonal and the planes $\alpha = g$, $\beta = -h$, $\gamma = \pm k$. The quantity in brackets in the first of expressions (6) will then be

$$
\frac{1}{lmn\pi^3}\Big\{\int_0^g\int_{-h}^0\int_{-k}^k + \int_0^g\int_0^{h\alpha/g}\int_{-k}^k + \int_{-g}^0\int_{-h}^{h\alpha/g}\int_{-k}^k\Big\},
$$

where the expression under the integral signs is the same as in (6). From Lemma 3 the first term of this expression approaches $\frac{1}{4}$ as l, m and n become infinite. If in the third term we make the change of variable

$$
\alpha = \alpha', \qquad \beta = \beta',
$$

this term combines with the second to produce an expression which, from Lemma 3, also approaches $\frac{1}{4}$ as l, m and n become infinite, and hence for this case also the first limit in (6) exists and is equal to $\frac{1}{2}$.

Hence this is true for all positions of the plane through the diagonal, and as shown before, the second limit in (6) must also exist and be equal to $\frac{1}{2}$ for all positions of the plane. The lemma is therefore proved.

LEMMA 5. *If R is the region described in Lemma 2, the limit*

$$\lim_{l,m,n\to\infty}\left[\frac{1}{lmn}\int\int\int_R \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma \right.$$
$$\left. -\frac{1}{lmn}\int\int\int_R \frac{\sin^2 l\alpha}{\alpha^2}\,\frac{\sin^2 m\beta}{\beta^2}\,\frac{\sin^2 n\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma\right]$$

will exist and be equal to zero.

The expression in brackets may be written in the form

$$\frac{1}{lmn}\int\int\int_R\left(1-\frac{\sin^2\alpha}{\alpha^2}\right)\left(1-\frac{\sin^2\beta}{\beta^2}\right)\left(1-\frac{\sin^2\gamma}{\gamma^2}\right)$$
$$\cdot\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma.$$

It may easily be seen that the product of the three terms in parentheses under the integral sign satisfies the condition imposed on $\varphi(\alpha,\beta,\gamma)$ in Lemma 2, and hence the lemma follows at once.

LEMMA 6. *Let R be a region within the cube described in Lemma 1, and without the cube $-\varepsilon\leqq\alpha\leqq\varepsilon$, $-\varepsilon\leqq\beta\leqq\varepsilon$, $-\varepsilon\leqq\gamma\leqq\varepsilon$, where ε is an arbitrarily small positive quantity and $\varphi(\alpha,\beta,\gamma)$ a function that is integrable in R and remains finite in that part of R which includes all points whose coördinates satisfy one or two of the conditions*

$$(8)\qquad -\varepsilon\leqq\alpha\leqq\varepsilon,\qquad -\varepsilon\leqq\beta\leqq\varepsilon,\qquad -\varepsilon\leqq\gamma\leqq\varepsilon.$$

Then the limit

$$(9)\qquad \lim_{l,m,n\to\infty}\frac{1}{lmn}\int\int\int_R\varphi(\alpha,\beta,\gamma)\,\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma$$

will exist and be equal to zero.

The quantity in (9) may be written in the form

$$(10)\qquad \frac{1}{lmn}\left\{\int\int\int_{R'}+\int\int\int_{R''}\right\}$$

where the integrands are the same as that in (9), R' that part of R which includes all points whose coördinates satisfy one or two of the conditions (8), and R'' the remainder of R.

By Lemma 1 the first term of (10) approaches zero as a limit as l, m and n become infinite.

Consider now the second term of (10):

$$\left| \frac{1}{lmn} \int\int\int_{R'} \varphi(\alpha, \beta, \gamma) \frac{\sin^2 l\alpha}{\sin^2 \alpha} \frac{\sin^2 m\beta}{\sin^2 \beta} \frac{\sin^2 n\gamma}{\sin^2 \gamma} d\alpha\, d\beta\, d\gamma \right|$$

$$< \frac{1}{lmn \sin^6 \eta} \int\int\int_{R'} |\varphi(\alpha, \beta, \gamma)|\, d\alpha\, d\beta\, d\gamma$$

where η is the smaller of the quantities ε and δ, δ being the least distance between R' and the cube $\alpha = \pm \pi$, $\beta = \pm \pi$, $\gamma = \pm \pi$. The right-hand side of this inequality may be made as small as we please by choosing l, m and n sufficiently large and therefore the second term of (10) approaches zero as l, m and n become infinite. Hence the limit (9) exists and is equal to zero and the lemma is proved.

Before proving the next theorem we shall define an expression which we wish to use in the statement of this and succeeding theorems. *A critical region associated with the point* (x_1, y_1, z_1) is defined as a region made up of points whose coördinates satisfy one or more of the inequalities

$$x_1 - \varepsilon \leqq x \leqq x_1 + \varepsilon, \quad y_1 - \varepsilon \leqq y \leqq y_1 + \varepsilon, \quad z_1 - \varepsilon \leqq z \leqq z_1 + \varepsilon,$$

where ε is an arbitrarily small positive quantity.

We are now ready to prove the following theorem:

THEOREM I. *If $f(x, y, z)$ is integrable in the region*

$$(11) \qquad (-\pi \leqq x \leqq \pi, \quad -\pi \leqq y \leqq \pi, \quad -\pi \leqq z \leqq \pi)$$

and (x_1, y_1, z_1) is a point of discontinuity of $f(x, y, z)$ such that every other point of discontinuity in the neighborhood of (x_1, y_1, z_1) lies on a plane passing through that point, and the function approaches a definite value as we approach the point from either side of the plane, the series (4) *will be summable $(C\,1)$ at that point to a value half way between the limiting values of the function, provided $f(x, y, z)$ remains finite in some critical region associated with the point (x_1, y_1, z_1).*

For the series (4) at the point (x_1, y_1, z_1) we have

$$
(12)\qquad \begin{aligned} \frac{S_{lmn}^{(1)}(x_1, y_1, z_1)}{lmn} &= \frac{1}{8\,lmn\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x', y', z') \\ &\quad \cdot \frac{\sin^2\dfrac{l(x'-x_1)}{2}}{\sin^2\dfrac{x'-x_1}{2}}\,\frac{\sin^2\dfrac{m(y'-y_1)}{2}}{\sin^2\dfrac{y'-y_1}{2}}\,\frac{\sin^2\dfrac{n(z'-z_1)}{2}}{\sin^2\dfrac{z'-z_1}{2}}\,dx'\,dy'\,dz', \end{aligned}
$$

where $S_{lmn}^{(1)}$ is defined by equation (1). Making the change of variable

$$
(13)\qquad x'-x_1 = 2\alpha, \qquad y'-y_1 = 2\beta, \qquad z'-z_1 = 2\gamma
$$

we have

$$
(14)\qquad \begin{aligned} \frac{S_{lmn}^{(1)}(x_1,y_1,z_1)}{lmn} &= \frac{1}{lmn\pi^3}\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_{-\frac{\pi+y_1}{2}}^{\frac{\pi-y_1}{2}}\int_{-\frac{\pi+z_1}{2}}^{\frac{\pi-z_1}{2}} f(x_1+2\alpha, y_1+2\beta, z_1+2\gamma) \\ &\quad \cdot \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma. \end{aligned}
$$

In order to prove the theorem we must show that as l, m and n become infinite, the right-hand side of the expression (14) is equal to one-half the sum of the limiting values of the function.

Since all the points of discontinuity in the neighborhood of (x_1, y_1, z_1) lie in a plane, we can determine a rectangular parallelepiped of dimensions $2g$, $2h$, $2k$ whose sides are parallel to the coördinate planes and such that the point (x_1, y_1, z_1) lies at the center of the parallelepiped and all other points of discontinuity lie on a plane through (x_1, y_1, z_1) either parallel to two opposite faces of the parallelepiped or passing through one of its diagonals.

If we make the transformation (13) this becomes a rectangular parallelepiped whose center lies at the point $\alpha=\beta=\gamma=0$. We may call this region R', designating the remainder of the region of integration in (14) by R''. We then have

$$
(15)\qquad \frac{S_{lmn}^{(1)}(x_1, y_1, z_1)}{lmn} = \frac{1}{lmn\pi^3}\int\int\int_{R'} + \frac{1}{lmn\pi^3}\int\int\int_{R''},
$$

where the quantity under the integral signs is understood to be the same as that in equation (14).

By virtue of Lemma 6 the second term of the right-hand side of this equation approaches zero as l, m and n become infinite.

The region R' is divided into two parts, R_1' and R_2', by the plane of discontinuity and hence the first term on the right-hand side of (15) may be written in the form

$$\frac{1}{lmn\pi^3}\int\int\int_{R_1'}+\frac{1}{lmn\pi^3}\int\int\int_{R_2'}, \tag{16}$$

the expression under the triple integral signs being the same as in equation (14).

If we represent by f_1 and f_2 the values which $f(x, y, z)$ approaches as we approach (x_1, y_1, z_1) through the regions R_1' and R_2' respectively, we may write the first term of (16) in the form

$$\frac{1}{lmn\pi^3}\int\int\int_{R_1'}\varphi(\alpha, \beta, \gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\, d\beta\, d\gamma$$

$$+\frac{f_1}{lmn\pi^3}\int\int\int_{R_1'}\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\, d\beta\, d\gamma,$$

where $\varphi(\alpha, \beta, \gamma) = f(x_1+2\alpha, y_1+2\beta, z_1+2\gamma)-f_1$. It then follows from Lemma 2 that the first term of this expression approaches zero as l, m and n become infinite. The second approaches $\frac{1}{2}f_1$ as l, m and n become infinite, by virtue of Lemma 3 if the plane of discontinuity is parallel to two opposite faces of the parallelepiped, or of Lemma 4 if it passes through a diagonal.

Similarly it may be shown that the second term of (16) approaches $\frac{1}{2}f_2$ as l, m and n become infinite and therefore (16), and hence the left-hand side of (14), approaches $\frac{1}{2}(f_1+f_2)$ as l, m and n become infinite and the theorem is proved.

For points on the boundaries of the region (11) except at the edges and vertices, the results are analogous to those for points in the interior and may be obtained by methods similar to those of Theorem I. These results are given in the following corollary:

COROLLARY. *If x_1 and y_1 are values of x and y lying in the interval $(-\pi < x < \pi,\ -\pi < y < \pi)$ and if $f(x, y, z)$ is integrable in the region* (11) *and furthermore is such that the limits*

$$\lim_{x\to x_1, y\to y_1, z\to \pi} f(x, y, z), \qquad \lim_{x\to x_1, y\to y_1, z\to -\pi} f(x, y, z) \tag{17}$$

exist, the Fourier development of $f(x, y, z)$ *will be summable* $(C1)$ *at the points* (x_1, y_1, π) *and* $(x_1, y_1, -\pi)$ *and its value will be one-half the sum of the two limits* (17), *provided* $f(x, y, z)$ *remains finite in some critical regions associated with* (x_1, y_1, π) *and* $(x_1, y_1, -\pi)$. *Analogous statements with corresponding conditions may be made for points on the other faces.*

We shall next consider some of the results obtained when (x_1, y_1, z_1) is a point of discontinuity of $f(x, y, z)$ such that all other points of discontinuity of the function in the neighborhood of that point lie on two planes through the point. When the planes are parallel to two of the coördinate planes the results are simple; this case, which is of special importance since it is analogous to the case where the points lie on the edges of the region of periodicity, is considered in the following theorem:

THEOREM II. *If* $f(x, y, z)$ *is integrable in the region* (11) *and* (x_1, y_1, z_1) *is a point of discontinuity of* $f(x, y, z)$ *such that every other point of discontinuity in its neighborhood lies on one of two planes which are parallel to two coördinate planes and pass through* (x_1, y_1, z_1), *and if furthermore the function approaches a definite value as we approach the point through each of the four regions into which the planes of discontinuity divide the neighborhood of that point, the series* (4) *will be summable* $(C1)$ *at* (x_1, y_1, z_1) *to a value which is one-fourth the sum of the four limiting values of the function, provided* $f(x, y, z)$ *remains finite in some critical region associated with the point* (x_1, y_1, z_1).

The proof of this theorem is similar to that of Theorem I. The following corollary may also be established by similar methods:

COROLLARY. *If* x_1 *is a value of* x *lying in the interval* $(-\pi < x < \pi)$ *and if* $f(x, y, z)$ *is integrable in the region* (11) *and furthermore is such that the four limits*

$$\lim_{x \to x_1,\, y \to \pm\pi,\, z \to \pm\pi} f(x, y, z) \tag{18}$$

exist, the Fourier development of $f(x, y, z)$ *will be summable* $(C1)$ *at each of the four points* $(x_1, \pm\pi, \pm\pi)$ *and to a value which is one-fourth the sum of the limits* (18). *Similar statements may be made under analogous conditions for points on the other edges.*

Remark.—The conclusions of each of the first six lemmas hold equally well if instead of becoming infinite together, l, m and n become infinite in any other manner. Hence Theorems I and II and their accompanying corollaries hold however l, m and n become infinite.

In all other cases where the point of discontinuity is such that all the points of discontinuity in its neighborhood lie on two planes through the point, the value which the expression

$$\text{(19)} \qquad \frac{S_{lmn}^{(1)}(x_1, y_1, z_1)}{lmn}$$

approaches as l, m and n become infinite depends on the positions of the planes and also the manner in which l, m and n become infinite. Before considering some of these cases it will be necessary to obtain some preliminary results.

For the purposes of our work we may divide the parallelepiped of Lemma 4 into four parts in two ways, namely by two planes each of which passes through a diagonal joining two opposite vertices of the parallelepiped or by two planes one of which passes through a diagonal and the other passes through the center and is parallel to one of the faces. We may now state the lemma:

LEMMA 7. (*a*) *If we divide the parallelepiped of Lemma* 4 *into four parts* R_1, R_2, R_3, R_4 *by either of the above methods, and if each of the ratios* m/n, n/l, l/m *approaches a finite limit or becomes infinite as* l, m *and* n *become infinite, the limits*

$$\text{(20)} \quad \lim_{l,m,n\to\infty} \frac{1}{lmn}\int\int\int_{R_i} \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma \ (i=1,2,3,4)$$

will exist and will each approach a limit between 0 *and* $\pi^3/2$. *Moreover the sum of two limits in which the regions of integration are adjacent is* $\pi^3/2$, *and the limits for which the regions of integration are opposite are equal. If any of the ratios between* l, m *and* n *oscillates, the limits will oscillate between* 0 *and* $\pi^3/2$, *their values being subject to the same restriction as above.*

(*b*) *If the intersection of the planes is parallel to the* α- (β- *or* γ-) *axis the results are the same as above except that only the ratio* m/n (n/l *or* l/m) *affects the values of the limits, which are fixed if this ratio remains fixed and oscillate if it oscillates.*

We shall consider, first, part (*a*). That the value of each limit lies between 0 and $\pi^3/2$ or is equal to one of them, subject to the restriction that the sum of two limits in which the regions of integration are adjacent is equal to $\pi^3/2$, is obvious from Lemma 3 if the plane used as the dividing plane is parallel to a coördinate axis, or from Lemma 4 if it is not.

That the limits are the same for opposite regions of integration may be seen by making suitable changes of variable of the type $\alpha = -\alpha'$.

Each of the limits (20) depends on a limit of the type

$$\lim_{l,m,n\to\infty} \frac{1}{lmn}\int_0^g\int_0^{p\alpha}\int_0^{q\alpha+r\beta} \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma$$

which, if we apply Lemma 5 and make the change of variable

$$(21)\qquad l\alpha = \alpha', \quad m\beta = \beta', \quad n\gamma = \gamma',$$

reduces to the form

$$\lim_{l,m,n\to\infty} \int_0^{lg}\int_0^{\frac{m}{l}p\alpha}\int_0^{\frac{n}{l}q\alpha+\frac{n}{m}r\beta} \frac{\sin^2\alpha}{\alpha^2}\,\frac{\sin^2\beta}{\beta^2}\,\frac{\sin^2\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma .$$

This limit depends on the values of the ratios between l, m und n and hence when any of these ratios oscillates the value of the limit oscillates.

To prove the lemma for case (b) we may assume that the intersection of the planes is parallel to the α-axis. Each of the limits (20) depends on a limit of the type

$$\lim_{l,m,n\to\infty} \frac{1}{lmn}\int_0^{g}\int_0^{h}\int_0^{k\beta} \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma$$

which reduces to the form

$$(22)\qquad \int_0^{\infty}\frac{\sin 2\alpha}{\alpha^2}\left\{\int_0^{\infty}\frac{\sin^2\beta}{\beta^2}\left(\int_0^{r\beta}\frac{\sin^2\gamma}{\gamma^2}\,d\gamma\right)d\beta\right\}d\alpha$$

by Lemma 5 and the change of variable (21) if we set

$$r = \lim_{m,n\to\infty}\frac{n}{m}k .$$

The proof then follows as in part (a).

In this case we may obtain an exact value for the limits (20) since they depend on triple integrals of the type of (22), which we may evaluate without great difficulty. The double integral in brackets in (22) has been evaluated by C. N. Moore*, and applying his results we have for the value of (22)

$$\frac{\pi^3}{8}+\frac{\pi}{2}\left\{\frac{(r+1)\log(r+1)-2r\log r+(r-1)\log(r-1)}{4}\right.$$
$$\left.+\frac{(r-1)\log(r-1)-(r+1)\log(r+1)}{4r}-\left(\frac{1}{r}+\frac{1}{3^2r^3}+\frac{1}{5^2r^5}+\cdots\right)\right\}(r>1),$$

$$\frac{\pi^3}{16}\qquad (r=1),$$

$$\frac{\pi}{2}\left\{\frac{(1+r)\log(1+r)-2r\log r-(1-r)\log(1-r)}{4}\right.$$
$$\left.-\frac{(1-r)\log(1-r)+(1+r)\log(1+r)}{4r}+\left(r+\frac{1}{3^2}r^3+\frac{1}{5^2}r^5+\cdots\right)\right\}(0<r<1).$$

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* Loc. cit. p. 564.

We may now consider the summability of $f(x,y,z)$ at a point of discontinuity such that all other points of discontinuity in its neighborhood lie on two planes both of which are not parallel to a coördinate plane. The proofs may easily be carried out by methods similar to those of Theorem I except that Lemma 7 is used instead of Lemmas 3 and 4.

Let $f(x,y,z)$ be a function that is integrable throughout the region (11) and (x_1, y_1, z_1) a point of discontinuity of $f(x, y, z)$ such that all the discontinuities in its neighborhood lie on two planes which pass through that point and whose intersection is not parallel to a coördinate axis, and furthermore let $f(x, y, z)$ remain finite in some critical region associated with (x_1, y_1, z_1) and approach a definite value as we approach (x, y, z) through each of the regions into which the planes of discontinuity divide the neighborhood of that point. Then if l, m and n become infinite in such a manner that any of the ratios n/m, l/n, m/l oscillates, the value of (19) will oscillate between $\frac{1}{2}(f_1+f_3)$ and $\frac{1}{2}(f_2+f_4)$, where f_1, f_2, f_3, f_4 are the values which the function approaches, assigned in rotation. If the ratios become infinite or remain finite the expression (19) will approach a finite limit between $\frac{1}{2}(f_1+f_3)$ and $\frac{1}{2}(f_2+f_4)$ or equal to one of them.

If $f(x, y, z)$ and (x_1, y_1, z_1) satisfy the conditions of the preceding paragraph except that the intersection of the planes of discontinuity is parallel to one of the coördinate axes, the only ratio which affects the limit of the expression (19) is n/m, l/n, m/l, according as the intersection of the planes is parallel to the x-, y- or z-axis. If this ratio oscillates the value of the limit (19) will oscillate between $\frac{1}{2}(f_1+f_3)$ and $\frac{1}{2}(f_2+f_4)$, and if it approaches a finite limit or becomes infinite (19) will approach a limit between these values or equal to one of them. The exact value may be obtained by determining the values of the limits (20) from the expressions for the integral (22).

In particular we may derive the following theorem:

THEOREM III. *Let $f(x, y, z)$ be a function which is integrable in the region* (11) *and (x_1, y_1, z_1) a point of discontinuity of $f(x, y, z)$ such that every other point of discontinuity in its neighborhood lies on two mutually perpendicular planes which pass through (x_1, y_1, z_1) and whose intersection is parallel to one of the coördinate axes. Then if $f(x, y, z)$ remains finite in some critical region associated with (x_1, y_1, z_1) and approaches a definite value as we approach that point through each of the four regions into which the planes of discontinuity divide its neighborhood, the series* (4) *will be summable* $(C1)$ *at (x_1, y_1, z_1) to one-fourth the sum of the four limiting values of the function, provided the ratio n/m, l/n, or m/l, according as the intersection of the planes of discontinuity is parallel to the x-, y-, or z-axis, approaches unity as l, m and n become infinite.*

Results similar to those obtained in the above discussion hold where the discontinuities are distributed on a broken plane instead of on two intersecting planes.

We may also carry through discussions of the cases where the discontinuities lie on three or more planes intersecting at a point of discontinuity, but as these results are in general complicated we shall confine ourselves to stating one particular case from which we may deduce the behavior of the series at the vertices of the region (11). The results are contained in the following theorem and its corollary, the proofs being carried out by a method similar to that used in Theorem I:

THEOREM IV. *If $f(x, y, z)$ is integrable in the region* (11) *and* (x_1, y_1, z_1) *is a point of discontinuity of $f(x, y, z)$ such that every other point of discontinuity in its neighborhood lies on three planes through that point and parallel respectively to the coördinate planes, and if moreover $f(x, y, z)$ approaches a definite value as we approach (x_1, y_1, z_1) through each of the eight regions into which the planes divide the neighborhood of that point, the series* (4) *will be summable* ($C1$) *at the point (x_1, y_1, z_1) and to a value which is one-eighth the sum of the eight limiting values of the function, provided $f(x, y, z)$ remains finite in some critical region associated with the point (x_1, y_1, z_1).*

COROLLARY. *If $f(x,y,z)$ is integrable in the region* (11) *and if the eight limits*

$$\lim_{x\to\pm\pi,\, y\to\pm\pi,\, z\to\pm\pi} f(x, y, z) \tag{23}$$

exist, the series (4) *will be summable* ($C1$) *at the eight vertices of the region* (11) *and its value will be one-eighth the sum of the eight limits* (23), *provided the function remains finite in some critical region of each of the eight vertices.*

II. DISCONTINUITIES WHICH LIE ALONG CURVED SURFACES

We shall now consider the behavior of the series at a point of discontinuity such that all other points of discontinuity in its neighborhood lie on a curved surface through the point and such that the function $f(x, y, z)$ approaches a definite value as we approach the point of discontinuity from either side of the surface. This surface is assumed to have a tangent plane at the point of discontinuity under consideration and to be of such nature that any plane through the point intersects the surface in a finite number of curves or not at all.

Before proving our main theorem we shall have to prove several lemmas.

LEMMA 8. *The integral*

$$\int_0^\infty \int_0^\infty \int_0^{\lambda\alpha+\mu\beta} \frac{\sin^2\alpha}{\alpha^2}\,\frac{\sin^2\beta}{\beta^2}\,\frac{\sin^2\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma \tag{24}$$

converges for all values of λ and μ and represents a function of λ and μ, $\psi(\lambda, \mu)$, which is continuous for all values of λ and μ and approaches $\pi^3/8$ as a limit as λ and μ become infinite in the same or opposite directions.

We shall consider first the case where λ and μ are both positive. If we let

$$\varphi(\lambda, \mu, \alpha, \beta) = \int_0^{\lambda\alpha+\mu\beta} \frac{\sin^2\gamma}{\gamma^2}\, d\gamma$$

we may write the integral (24) in the form

$$\int_0^\infty \int_0^\infty \varphi(\lambda, \mu, \alpha, \beta) \frac{\sin^2\alpha}{\alpha^2} \frac{\sin^2\beta}{\beta^2}\, d\alpha\, d\beta. \tag{25}$$

It may readily be seen that $\varphi(\lambda, \mu, \alpha, \beta)$ is a continuous function of $\lambda, \mu, \alpha, \beta$ for all values of these arguments and that it remains finite for all such values. Therefore the integrand of (25) is a continuous function and as the double integral (25) is uniformly convergent, $\psi(\lambda, \mu)$, the function represented by (25), defines a function of λ and μ which is continuous for all values of λ and μ.

If λ and μ are of opposite sign, say λ is negative, we may make the substitution $\alpha = -\alpha'$ and the proof follows in the same manner as before. If both λ and μ are negative, we set $\alpha = -\alpha'$, $\beta = -\beta'$, and the proof follows similarly.

LEMMA 9. *The integral*

$$\int_0^\infty \int_0^{-\frac{\lambda}{\mu}\alpha} \int_0^{\lambda\alpha+\mu\beta} \frac{\sin^2\alpha}{\alpha^2} \frac{\sin^2\beta}{\beta^2} \frac{\sin^2\gamma}{\gamma^2}\, d\alpha\, d\beta\, d\gamma, \tag{26}$$

where μ remains fixed, converges for all values of λ and represents a function of λ, $\Omega(\lambda)$, which is continuous for all values of λ and remains finite as λ becomes positively or negatively infinite.

Defining

$$\int_0^{-\frac{\lambda}{\mu}\alpha} \int_0^{\lambda\alpha+\mu\beta} \frac{\sin^2\beta}{\beta^2} \frac{\sin^2\gamma}{\gamma^2}\, d\beta\, d\gamma$$

by $\chi(\lambda, \alpha)$, we may write the integral (26) in the form

$$\int_0^\infty \chi(\lambda, \alpha) \frac{\sin^2\alpha}{\alpha^2}\, d\alpha. \tag{27}$$

It is obvious that $\chi(\lambda, \alpha)$ is a continuous function of λ and α for all values of these arguments and that it remains finite for all such values. Hence the integrand of (27) is a continuous function of λ and α for all values of these arguments and since the integral is uniformly convergent for all values of λ, it represents a function, $\Omega(\lambda)$, which is continuous for all values of λ.

As λ becomes infinite $\chi(\lambda, \alpha)$ approaches the limit $\pi^2/4$, provided $\alpha \neq 0$, and hence $\Omega(\lambda)$ approaches $\pi^3/8$ as a limit and therefore remains finite for all positive values of λ.

For the case where λ is negative, the proof may be carried out by making the change of variable $\alpha = -\alpha'$.

LEMMA 10. *If $\gamma = G(\alpha, \beta)$ represents a curved surface whose tangent plane at the origin is $\gamma = \lambda\alpha + \mu\beta$ and whose intersection with the α, β plane is $\beta = g(\alpha)$, and if $\gamma = G(\alpha, \beta)$ is intersected in only a finite number of curves in the neighborhood of the origin by any plane passing through that point, the limits*

$$(28)\quad \lim_{l,m,n\to\infty}\left[\frac{1}{lmn}\int_0^h\int_0^k\int_0^{G(\alpha,\beta)}\frac{\sin^2\alpha}{\alpha^2}\,\frac{\sin^2\beta}{\beta^2}\,\frac{\sin^2\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma - \frac{1}{lmn}\int_0^h\int_0^k\int_0^{\lambda\alpha+\mu\beta}\frac{\sin^2\alpha}{\alpha^2}\,\frac{\sin^2\beta}{\beta^2}\,\frac{\sin^2\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma\right],$$

$$(29)\quad \lim_{l,m,n\to\infty}\left[\frac{1}{lmn}\int_0^h\int_0^{g(\alpha)}\int_0^{G(\alpha,\beta)}\frac{\sin^2\alpha}{\alpha^2}\,\frac{\sin^2\beta}{\beta^2}\,\frac{\sin^2\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma - \frac{1}{lmn}\int_0^h\int_0^{-\frac{\lambda}{\mu}\alpha}\int_0^{\lambda\alpha+\mu\beta}\frac{\sin^2\alpha}{\alpha^2}\,\frac{\sin^2\beta}{\beta^2}\,\frac{\sin^2\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma\right],$$

where h and k are constants between 0 *and* π, *will exist and be equal to zero provided that λ and μ are not zero in* (28) *and that μ is not zero in* (29).*

We shall carry out the proof where λ and μ are both positive, the other cases following by a change of variable as in Lemmas 8 and 9.

Consider first the expression (28), writing the quantity in brackets in the form

* If the equation of the surface or tangent plane is such that it can not be put in the form used in the statement of the lemma, we may use α or β as the left-hand side of the equations, modifying the remainder of the statement and the proof accordingly.

$$
(30)\qquad
\begin{aligned}
&\frac{1}{lmn}\left\{\int_0^p\int_0^q\int_0^{G(\alpha,\beta)}-\int_0^p\int_0^q\int_0^{\lambda\alpha+\mu\beta}\right\}\\
&+\frac{1}{lmn}\left\{\int_0^h\int_0^k\int_0^{G(\alpha,\beta)}-\int_0^p\int_0^q\int_0^{G(\alpha,\beta)}\right\}\\
&+\frac{1}{lmn}\left\{\int_0^p\int_0^q\int_0^{\lambda\alpha+\mu\beta}-\int_0^h\int_0^k\int_0^{\lambda\alpha+\mu\beta}\right\},
\end{aligned}
$$

where $0<p<h$, $0<q<k$, the expression under the triple integral signs being the same as that in (28). It is obvious that the second and third terms of (30) approach zero as l, m and n become infinite. Then if we can show that p and q may be so chosen that for sufficiently large values of l, m and n the first term of (30) may be made as small as we please in absolute value, the lemma will have been demonstrated for the expression (28).

Assume for the sake of definiteness that the portion of the surface above the α,β plane lies above the tangent plane until possibly it intersects it. We may choose a λ' and μ' greater respectively than λ and μ, such that $\lambda'/\mu'=\lambda/\mu$ and such that the plane $\gamma=\lambda'\alpha+\mu'\beta$ intersects the surface in a curve whose projection on the α,β plane lies entirely within the projection of any curve in which the tangent may cut the surface in the octant $0-XYZ$. If we draw a rectangle two of whose sides lie along the α and β axes, within the area included between these axes and the projection of the intersection of $\gamma=\lambda'\alpha+\mu'\beta$ and $\gamma=G(\alpha,\beta)$ in this octant, the vertex opposite the origin being $(p,q,0)$, we have

$$
(31)\qquad \frac{1}{lmn}\int_0^p\int_0^q\int_0^{\lambda'\alpha+\mu'\beta}>\frac{1}{lmn}\int_0^p\int_0^q\int_0^{G(\alpha,\beta)}>\frac{1}{lmn}\int_0^p\int_0^q\int_0^{\lambda\alpha+\mu\beta},
$$

the expression under the integral signs being the same as that in (28). Making the transformation (21) we see that, p and q being fixed, we can choose l and m so large that the first and third members of the inequality (31) become as near $\psi(n\lambda'/l,n\mu'/m)$ and $\psi(n\lambda/l,n\mu/m)$ respectively as we wish, this function being defined as in Lemma 8. But since by that lemma this function is continuous, it follows that for λ' and μ' sufficiently near to λ and μ respectively, $\psi(n\lambda'/l,n\mu'/m)$ and $\psi(n\lambda/l,n\mu/m)$ differ for all values of l, m and n by as small a quantity as we please.*

* The restriction $\lambda\neq 0$, $\mu\neq 0$ is of course essential for this conclusion. When these conditions do not hold, we must limit the freedom of variation of l, m and n as is pointed out in Lemma 11.

Hence from (31) p and q may be so chosen that for l, m and n large enough the first term of (30) is as small in absolute value as we please, and the lemma is proved for the expression (28).

To prove it for (29) we write the quantity in brackets in that expression in the form

$$
\begin{aligned}
&\frac{1}{lmn}\left\{\int_0^r\int_0^{g(\alpha)}\int_0^{G(\alpha,\beta)}-\int_0^r\int_0^{-\frac{\lambda}{\mu}\alpha}\int_0^{\lambda\alpha+\mu\beta}\right\}\\
(32)\qquad &+\frac{1}{lmn}\left\{\int_0^h\int_0^{g(\alpha)}\int_0^{G(\alpha,\beta)}-\int_0^r\int_0^{g(\alpha)}\int_0^{G(\alpha,\beta)}\right\}\\
&+\frac{1}{lmn}\left\{\int_0^r\int_0^{-\frac{\lambda}{\mu}\alpha}\int_0^{\lambda\alpha+\mu\beta}-\int_0^h\int_0^{-\frac{\lambda}{\mu}\alpha}\int_0^{\lambda\alpha+\mu\beta}\right\},
\end{aligned}
$$

where $0 < r < h$, the expression under the triple integral signs being the same as that in (29).

Here as in (30) the second and third terms approach zero as l, m and n become infinite. Then we have only to show that r may be so chosen that for sufficiently large values of l, m and n the first term of (32) may be made as small in absolute value as we please, and the lemma will have been proved for the expression (29).

We shall again use the surface which we used in the proof for the expression (28), considering that part which lies in the octant $0-XY'Z$. We may then choose a λ' greater than λ and such that the plane $\gamma = \lambda'\alpha + \mu\beta$ intersects the surface in a curve whose projection on the α, β plane in the octant $0-XY'Z$ lies entirely within the projection of any curve in which the tangent plane may cut the surface in that octant. Then if r is the α coördinate of the first point to the right of the origin in which the tangent plane intersects the surface, we have

$$
\begin{aligned}
&\frac{1}{lmn}\int_0^r\int_{-\frac{\lambda'}{\mu}\alpha}^0\int_0^{\lambda'\alpha+\mu\beta} > \frac{1}{lmn}\int_0^r\int_{g(\alpha)}^0\int_0^{G(\alpha,\beta)}\\
(33)\qquad &\qquad > \frac{1}{lmn}\int_0^r\int_{-\frac{\lambda}{\mu}\alpha}^0\int_0^{\lambda\alpha+\mu\beta},
\end{aligned}
$$

where the expression under the integral signs is the same as before.

Making the transformation (21) we proceed by a method analogous to that used in the proof for the expression (28), using Lemma 9 instead of Lemma 8, and the lemma is proved for the expression (29).

LEMMA 11. *If all the conditions of Lemma* 10 *are satisfied by* $\gamma = G(\alpha, \beta)$ *except that the tangent to the surface passes through the* α- *or* β-*axis or coincides with the* α, β *plane we have the following results*:

(*a*) *if the tangent plane passes through the* α-*axis* $(\lambda = 0)$, *the limit* (28) *exists and is equal to zero provided* n *and* l *become infinite in such a manner that*

$$n/l < K, \tag{34}$$

where K *is a positive constant. Similarly if the tangent passes through the* β-*axis* $(\mu = 0)$, *the limit* (28) *exists and is equal to zero provided that* n *and* m *become infinite in such a manner that*

$$n/m < K;$$

(*b*) *if the tangent plane passes through the* α-*axis the limit*

$$\lim_{l,m,n\to\infty} \frac{1}{lmn} \int_0^h \int_{g(\alpha)}^0 \int_0^{G(\alpha,\beta)} \frac{\sin^2\alpha}{\alpha^2} \frac{\sin^2\beta}{\beta^2} \frac{\sin^2\gamma}{\gamma^2}\, d\alpha\, d\beta\, d\gamma$$

will exist and be equal to zero provided l, m *and* n *become infinite in such a way that the ratio between any two of them remains finite*;

(*c*) *if the tangent plane coincides with the* α, β *plane the limit*

$$\lim_{l,m,n\to\infty} \frac{1}{lmn} \int_0^h \int_0^k \int_0^{G(\alpha,\beta)} \frac{\sin^2\alpha}{\alpha^2} \frac{\sin^2\beta}{\beta^2} \frac{\sin^2\gamma}{\gamma^2}\, d\alpha\, d\beta\, d\gamma$$

will be exist and be equal to zero provided l, m *and* n *become infinite as provided in* (*b*).

The proofs involved in this lemma may be obtained by making slight changes in Lemma 10. For example, in the first part of (*a*) where $\lambda = 0$, we cannot say that if λ' is taken near enough to λ, $\psi(n\lambda'/l, n\mu'/m)$ and $\psi(n\lambda/l, n\mu/m)$ differ by as small a quantity as we please for all values of l, m and n. The statement is true for only such values of l and n as satisfy the inequality (34). The remaining proofs may be carried out by making similar changes in Lemma 10.

Before proceeding with the next theorem we shall define a term which we wish to use in the statement of the theorem. If the quotient in (3), corresponding to a triple series $\sum a_{lmn}$, approaches a limit as l, m and n

become infinite provided the ratios between l, m and n remain less in absolute value than a positive constant, we say that the triple series is *restrictedly summable* (Cr) and that its limit is the value approached. If only one ratio needs to be restricted we say that the series is *restrictedly summable with respect to the quantities involved in that ratio.*

THEOREM V. (*a*) *Let* $f(x, y, z)$ *be a function which is integrable throughout the region* (11) *and* (x_1, y_1, z_1) *a point of discontinuity of* $f(x, y, z)$ *such that every other point of discontinuity in its neighborhood lies on a surface satisfying the following conditions*: (1) *the surface has a tangent plane at the point* (x_1, y_1, z_1) *which is not parallel to any of the coördinate axes*; (2) *no plane through the point* (x_1, y_1, z_1) *intersects the surface in an infinite number of curves in the neighborhood of that point*; *and* (3) *the function* $f(x, y, z)$ *approaches a definite value as we approach* (x_1, y_1, z_1) *from either side of the surface. Then the series* (4) *will be summable* $(C1)$ *at the point* (x_1, y_1, z_1) *to a value half way between the limiting values of the function at that point, provided* $f(x, y, z)$ *remains finite in some critical region associated with* (x_1, y_1, z_1).

(*b*) *If all the other conditions are fulfilled, but the tangent plane to the surface at* (x_1, y_1, z_1) *is parallel to a coördinate axis or plane, the series will be restrictedly summable to one-half the sum of the limiting values of the function at* (x_1, y_1, z_1). *If, however, the plane tangent to the surface at* (x, y, z) *is parallel to only one of the coördinate axes and remains tangent to the surface in a line parallel to that axis, only one ratio needs to be restricted, the ratio being* n/m, l/n, m/l, *according as the line of tangency is parallel to the* x-, y-, *or* z-*axis.*

We shall consider first the case (*a*). We may choose a rectangular parallelepiped R with (x_1, y_1, z_1) as its center, such that the tangent plane passes through one diagonal and all points of discontinuity in the parallelepiped lie on the surface of discontinuity, and small enough so that the surface of discontinuity does not intersect the plane tangent to the surface within the parallelepiped, except perhaps at the point of tangency.

The Fourier series corresponding to $f(x, y, z)$ may then be written

$$
\begin{aligned}
\frac{S^{(1)}_{lmn}(x_1, y_1, z_1)}{lmn} &= \frac{1}{lmn\pi^3}\iiint_{R'} f(x_1+2\alpha, y_1+2\beta, z_1+2\gamma) \\
&\qquad\cdot\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma \\
&\quad+\frac{1}{lmn\pi^3}\iiint_{R''} f(x_1+2\alpha, y_1+2\beta, z_1+2\gamma) \\
&\qquad\cdot\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma,
\end{aligned}\tag{35}
$$

where R' represents the rectangular parallelepiped into which R is transformed by the change of variable (13) and R'' is the remainder of the region of integration of the integral (14). The second term of (35) vanishes as l, m and n become infinite, by Lemma 1. Then in order to prove our theorem we have only to show that as l, m and n become infinite the first term approaches a value which is one-half the sum of the limiting values of $f(x, y, z)$ at (x_1, y_1, z_1).

Let the surface of discontinuity divide the parallelepiped into two parts, R_1' and R_2' and let f_1 and f_2 be the limiting values of the function as we approach the point of discontinuity through R_1' and R_2' respectively. Then the first term of (35) may be written

$$\frac{1}{lmn\pi^3}\int\int\int_{R_1'} + \frac{1}{lmn\pi^3}\int\int\int_{R_2'} \tag{36}$$

where the quantity under the triple integral signs is the same as in (35). We shall show that the first term of (36) approaches $\frac{1}{2}f_1$ as a limit as l, m and n become infinite.

The tangent plane to the surface at (x_1, y_1, z_1) divides the region R_1' into two parts; let D_1 be that one which lies on the same side of the tangent plane as R_1' does of the surface of discontinuity. Then by Lemma 10

$$\begin{aligned} &\frac{1}{lmn}\int\int\int_{D_1} \frac{\sin^2 l\alpha}{\alpha^2}\,\frac{\sin^2 m\beta}{\beta^2}\,\frac{\sin^2 n\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma \\ &-\frac{1}{lmn}\int\int\int_{R_1'} \frac{\sin^2 l\alpha}{\alpha^2}\,\frac{\sin^2 m\beta}{\beta^2}\,\frac{\sin^2 n\gamma}{\gamma^2}\,d\alpha\,d\beta\,d\gamma \end{aligned} \tag{37}$$

approaches zero as l, m and n become infinite, and hence by Lemma 5

$$\begin{aligned} &\frac{1}{lmn}\int\int\int_{D_1} \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma \\ &-\frac{1}{lmn}\int\int\int_{R_1'} \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma \end{aligned} \tag{38}$$

approaches zero as l, m and n become infinite. But by Lemma 4 the first term of (37) approaches $\pi^3/2$ and hence the second term must also approach $\pi^3/2$ as l, m and n become infinite. Therefore the first term of (36) approaches $\frac{1}{2}f_1$ as l, m and n become infinite. Similarly it may be shown that the second

term of (36) approaches the value $\frac{1}{2}f_2$ and therefore the whole expression (36) and hence the right-hand side of (35) approaches $\frac{1}{2}(f_1+f_2)$ as l, m and n become infinite, and part (a) of the theorem is proved. Part (b) may be proved similarly by using Lemma 10 instead of Lemma 9.

We wish now to show that restricted summability is all that may be established for part (b). Since the proofs are analogous for the cases where the tangent plane is parallel to a coördinate axis or to a coördinate plane, we shall give the proof only for the latter, considering the development of a function that is continuous throughout the region (11) except on a curved surface lying above the x, y plane and tangent to it at the origin, and that approaches f_1 as we approach the surface of discontinuity from above the surface and f_2 as we approach it from below.

From Moore's generalization of Fejér's theorem,

$$f(x',0,0) = \sum_{m=1,n=1}^{\infty,\infty} \frac{1}{2^{E(1/m)+E(1/n)}\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x',y',z') \cdot \cos[(m-1)y']\cos[(n-1)z']\,dx'dy'dz' \quad (x' \neq 0).$$

The series on the right may be integrated term by term if we multiply it by a continuous function of x, and therefore

$$\frac{1}{2^{E(1/l)}\pi}\int_{-\pi}^{\pi} f(x',0,0)\cos[(l-1)x']\,dx' = \sum_{m=1,n=1}^{\infty,\infty} \frac{1}{2^{E(1/l)+E(1/m)+E(1/n)}\pi^3} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x',y',z') \cdot \cos[(l-1)x']\cos[(m-1)y']\cos[(n-1)z']\,dx'dy'dz',$$

and by the generalization of Fejér's theorem,

$$\sum_{l=1}^{\infty}\left\{\sum_{m=1,n=1}^{\infty,\infty} \frac{1}{2^{E(1/l)+E(1/m)+E(1/n)}\pi^3} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x',y',z') \cdot \cos[(l-1)x']\cos[(m-1)y']\cos[(n-1)z']\,dx'dy'dz'\right\} = \sum_{l=1}^{\infty}\frac{1}{2^{E(1/l)}\pi}\int_{-\pi}^{\pi} f(x',0,0)\cos[(l-1)x']\,dx' = f_2,$$

and the expression $(1/l\,m\,n)\,[S^{(1)}_{lmn}(0,0,0)]$ approaches f_2 if we let m and n become infinite holding l fixed, and then let l become infinite. If we let l and n become infinite holding m fixed and then let m become infinite the series is again summable to f_2 at the origin, but if we let l and m become infinite and then n, the value to which the series is summable at the origin is $\frac{1}{2}(f_1+f_2)$ and hence the triple series cannot be summable in the ordinary sense. As pointed out above, it may be shown in analogous fashion that restricted summability is all that can be proved when the tangent plane is parallel to one coördinate axis.

By methods similar to those of Theorem V and its preliminary lemmas we may obtain the value to which the function is summable at points of discontinuity such that all other points of discontinuity in their neighborhood are on two or more curved surfaces through the point, or lie at the intersection of two or more such surfaces. The results in these cases are analogous to those of Theorems II, III and IV, except that when the tangent to one or more of the curved surfaces is parallel to a coördinate axis we have restricted instead of ordinary summability.

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AN UNUSUAL TYPE OF EXPANSION PROBLEM*

BY

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In the majority of papers on the expansion problems associated with linear differential systems or with integral equations, the chief aim is a proof that a function, arbitrary within the limits of certain restrictions as light as possible, can be expanded in a uniformly convergent series of functions arising from the differential system or the integral equation. As exceptions to the rule we may cite a number of papers dealing with differential systems for which the boundary conditions are of irregular type.† In the following paragraphs we present a study of the linear differential system of the first order, without the customary limitation that a certain coefficient in the differential equation remain positive; the results obtained are in sharp contrast not only with those found for the usual expansion problems and those already mentioned in connection with irregular boundary conditions, but also with the facts for the precisely analogous second order differential system.‡ A second point of some interest is the appearance of a normal orthogonal set of trigonometric functions differing from the normalized Fourier set but as closely related to it as are the sine and cosine sets. It is interesting that this exceedingly simple differential system seems to have escaped attention entirely. So far as we know, the existence of this gap in the theory of boundary value and expansion problems was first pointed out by Professor Birkhoff in a recent course of lectures on the theory of linear differential equations.

I. The formal expansion problem

We shall study here the two linear differential systems

$$(1) \qquad u' + \lambda p u = 0, \qquad 0 \leqq x \leqq 1; \qquad u(0) - u(1) = 0;$$

$$(2) \qquad u' + \lambda p u = 0, \qquad 0 \leqq x \leqq 1; \qquad u(0) + u(1) = 0;$$

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† Jackson, Proceedings of the American Academy of Arts and Sciences, vol. 51 (1915–16), pp. 383–417; Hopkins, these Transactions, vol. 20 (1919), pp. 245–259.

‡ Mason, these Transactions, vol. 8 (1907), pp. 427–432; Lichtenstein, Rendiconti del Circolo Matematico di Palermo, vol. 38 (1914), pp. 113–166.

in which λ is a complex parameter and p is a bounded summable function with Lebesgue integral $\int_0^1 p\,dx$ equal to unity, which satisfies the further requirement that a finite set of non-overlapping intervals $\varrho_1, \varrho_2, \cdots, \varrho_n$ can be determined such that

(a) the set covers the unit interval just once;

(b) on ϱ_i either $p > 0$ almost everywhere (i.e., except upon a set of points of zero measure) or $p < 0$ almost everywhere;

(c) if $p > 0$ on ϱ_i almost everywhere, then $p < 0$ almost everywhere on the two adjacent ϱ-intervals and vice versa.

It has been customary to assume $p \geqq l > 0$ on the unit interval in discussions of expansion problems; in the present paper we dispense with this requirement. It will be apparent in the course of the development that there is no essential generalization in supposing p of the character already described rather than continuous with a finite number of changes of sign. At this point we may also remark that the more general differential system

$$u' + (\lambda p + q)u = 0, \qquad a \leqq x \leqq b,$$
$$Au(a) + Bu(b) = 0,$$

where λ and p are restricted as before except that $\int_0^1 p\,dx$ is merely different from zero, q is any summable function, and A and B are real numbers different from zero, can be reduced by a series of real linear transformations of the independent variable and of the parameter together with a real transformation of the independent variable in the form

$$u = u_1 A(x_1), \qquad A(x_1) \neq 0, \qquad 0 \leqq x_1 \leqq 1,$$

to either (1) or (2) according as $AB < 0$ or $AB > 0$; the transformation can be effected in the sense that the new differential equation is true except possibly on a set of zero measure. If complex transformations are admitted (1) can be taken into (2), and conversely, in an analogous manner.

The adjoint differential systems for (1) and (2) are readily computed as

$$-v' + \lambda p v = 0, \qquad -v(0) + v(1) = 0; \tag{3.1}$$

$$-v' + \lambda p v = 0, \qquad -v(0) - v(1) = 0. \tag{3.2}$$

The differential equation $u' + \lambda p u = 0$ has the function

$$u = e^{-\lambda P(x)}, \qquad P(x) = \int_0^x p\,dx,$$

as a solution in the sense that u satisfies the differential equation except possibly on a set of measure zero. That this is the case may easily be verified by substitution, if it is remembered that $P'(x) = p(x)$ almost everywhere, in accord with the theory of Lebesgue integration.* Furthermore, any other solution u_1 which is a Lebesgue integral is a constant multiple of u. To prove this we notice that

$$\frac{d}{dx}\left(\frac{u_1}{u}\right) = e^{\lambda P(x)}[u_1' + \lambda p u_1] = 0.$$

On integration we find that $u_1/u = a\ constant$. In the same way, the adjoint differential equation $-v' + \lambda p v = 0$ has a solution $v = e^{\lambda P(x)}$, unique except for a constant factor. When these solutions u and v are substituted in the boundary conditions of the two differential systems and in the adjoint boundary conditions, the characteristic equations

$$e^{\lambda} = 1, \tag{4.1}$$

$$e^{\lambda} = -1 \tag{4.2}$$

are obtained. The roots of these equations, the characteristic numbers, are immediately seen to be

$$\lambda_k = 2k\pi i, \qquad k = 0, \pm 1, \pm 2, \cdots; \tag{5.1}$$

$$\lambda_k = \pi i + 2k\pi i, \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{5.2}$$

In case we had admitted the equality $\int_0^1 p\,dx = 0$ we should have found here that every λ is a characteristic value in (1), and that no λ is a characteristic value in (2). Under our hypotheses the characteristic values form the discrete sets of numbers just determined. For solutions of (1) and (3.1) corresponding to distinct characteristic numbers λ_j, λ_k, we have, from the differential equations,

$$(\lambda_j - \lambda_k)\int_0^1 p u_k v_j\,dx = \int_0^1 (u_k v_j' + u_k' v_j)\,dx = u_k(1)v_j(1) - u_k(0)v_j(0).$$

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* Lebesgue, *Leçons sur l'Intégration*, Paris, 1904, pp. 124–125; de la Vallée Poussin, *Intégrales de Lebesgue*, Paris, 1916, § 67.

On account of the boundary conditions for (1) and (3.1) the last expression vanishes, so that

$$\int_0^1 p u_k v_j \, dx = 0.$$

On the other hand, for solutions corresponding to the same characteristic number we are led to

$$\int_0^1 p u_k v_k \, dx = \int_0^1 p e^{-\lambda_k P(x)} e^{+\lambda_k P(x)} \, dx = \int_0^1 p \, dx = 1.$$

In similar fashion we find for (2) and (3.2)

$$\int_0^1 p u_k v_j \, dx = \begin{Bmatrix} 0 & k \neq j \\ 1 & k = j \end{Bmatrix}.$$

By analogy with Fourier series the following formal expansions of an arbitrary function integrable in the sense of Lebesgue are suggested:

$$\text{(6.1)} \qquad \sum_{-\infty}^{+\infty} a_k u_k, \qquad a_k = \int_0^1 p f v_k \, dx;$$

$$\text{(6.2)} \qquad \sum_{-\infty}^{+\infty} a_k u_k, \qquad a_k = \int_0^1 p f v_k \, dx;$$

$$\text{(7.1)} \qquad \sum_{-\infty}^{+\infty} b_k v_k, \qquad b_k = \int_0^1 p f u_k \, dx;$$

$$\text{(7.2)} \qquad \sum_{-\infty}^{+\infty} b_k v_k, \qquad b_k = \int_0^1 p f u_k \, dx.$$

If we perform the formal operation of collecting the conjugate complex terms in these series, we find two related formal series of real terms with real coefficients. For the series (6.1) and (7.1) the terms with subscripts $+k$ and $-k$ respectively are conjugate complex; and the grouping of these terms yields one real series

$$\text{(8.1)} \qquad A_0 + \sum_1^{\infty} \{ A_k \sqrt{2} \cos 2k\pi P(x) + B_k \sqrt{2} \sin 2k\pi P(x) \};$$

$$A_0 = \int_0^1 p f \, dx, \qquad A_k = \int_0^1 p f \sqrt{2} \cos 2k\pi P(x) \, dx,$$

$$B_k = \int_0^1 p f \sqrt{2} \sin 2k\pi P(x) \, dx.$$

Similarly, the pairs of conjugate complex terms in the series (6.2) and (7.2) are those with subscripts $k \geqq 0$, $-k-1$ respectively; and the series obtained by collecting terms is

$$(8.2) \quad \sum_{0}^{\infty} \{ A_k \sqrt{2} \cos(2k+1)\pi P(x) + B_k \sqrt{2} \sin(2k+1)\pi P(x) \};$$

$$A_k = \int_0^1 pf \sqrt{2} \cos(2k+1)\pi P(x)\, dx,$$

$$B_k = \int_0^1 pf \sqrt{2} \sin(2k+1)\pi P(x)\, dx.$$

It should be noted that the set of functions

$$1, \quad \sqrt{2}\cos 2\pi x, \quad \sqrt{2}\sin 2\pi x, \quad \sqrt{2}\cos 4\pi x, \quad \sqrt{2}\sin 4\pi x, \cdots,$$

which serves as a basis for the formation of the series (8.1), is merely the ordinary Fourier set transformed to the unit interval and normalized.

II. The case $p \equiv 1$

Before proceeding to more general considerations we propose to discuss the series (8.1) and (8.2) when $p \equiv 1$ almost everywhere. In this case $P(x) \equiv x$ so that the expansions appear as series formed from the two normal orthogonal sets

$$(9) \quad 1, \quad \sqrt{2}\cos 2\pi x, \quad \sqrt{2}\sin 2\pi x, \quad \sqrt{2}\cos 4\pi x, \quad \sqrt{2}\sin 4\pi x, \cdots,$$

$$(10) \quad \sqrt{2}\cos \pi x, \quad \sqrt{2}\sin \pi x \quad \sqrt{2}\cos 3\pi x, \quad \sqrt{2}\sin 3\pi x, \cdots,$$

on the interval (0,1). Concerning the expansions of an arbitrary function integrable in the sense of Lebesgue we shall prove the

Theorem I. *The formal expansions of an arbitrary summable function in terms of the sets* (9) *and* (10) *on the interval* (0,1) *are the formal expansions in terms of the normalized Fourier set for the interval* (0,2) *of two functions* f_1 *and* f_2 *respectively.*

Proof. The normalized Fourier set for the interval (0,2) is seen to be

$$(11) \quad \frac{1}{\sqrt{2}}, \quad \cos\pi x, \quad \sin\pi x, \quad \cos 2\pi x, \quad \sin 2\pi x, \quad \cos 3\pi x, \quad \sin 4\pi x, \cdots.$$

If we denote by a_k, b_k the Fourier coefficients of a summable function in terms of (11) we have

$$
\begin{aligned}
a_0 &= \frac{1}{\sqrt{2}}\int_0^2 f\,dx &&= \frac{1}{\sqrt{2}}\int_0^1 \{f(x)+f(x+1)\}\,dx,\\
a_{2k} &= \int_0^2 f\cos 2k\pi x\,dx &&= \int_0^1 \{f(x)+f(x+1)\}\cos 2k\pi x\,dx,\\
b_{2k} &= \int_0^2 f\sin 2k\pi x\,dx &&= \int_0^1 \{f(x)+f(x+1)\}\sin 2k\pi x\,dx,\\
a_{2k+1} &= \int_0^2 f\cos(2k+1)\pi x\,dx &&= \int_0^1 \{f(x)-f(x+1)\}\cos(2k+1)\pi x\,dx,\\
b_{2k+1} &= \int_0^2 f\sin(2k+1)\pi x\,dx &&= \int_0^1 \{f(x)-f(x+1)\}\sin(2k+1)\pi x\,dx.
\end{aligned}
$$

If there is given an arbitrary summable function $f(x)$, $0 \leqq x \leqq 1$, we define $f_1(x)$ for the interval (0,2) by the identities

$$
\begin{aligned}
f_1(x) &\equiv f(x), &\quad 0 \leqq x \leqq 1,\\
f_1(x+1) &\equiv f(x), &\quad 0 < x \leqq 1.
\end{aligned}
$$

Then the Fourier series of f_1 on (0,2) has the coefficients

$$
a_0 = \sqrt{2}\int_0^1 f\,dx,\quad a_{2k} = 2\int_0^1 f\cos 2k\pi x\,dx,\quad b_{2k} = 2\int_0^1 f\sin 2k\pi x\,dx,
$$

$$
a_{2k+1} = b_{2k+1} = 0.
$$

Furthermore the Fourier series thus set up,

$$
\frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty}\{a_{2k}\cos 2k\pi x + b_{2k}\sin 2k\pi x\},
$$

is term for term the formal series for $f(x)$ in terms of (9) on (0,1).

Similarly, if we take $f_2(x)$ such that

$$
\begin{aligned}
f_2(x) &\equiv f(x), &\quad 0 \leqq x \leqq 1,\\
f_2(x+1) &\equiv -f(x), &\quad 0 < x \leqq 1,
\end{aligned}
$$

there results

$$
a_0 = a_{2k} = b_{2k} = 0,
$$

$$
a_{2k+1} = 2\int_0^1 f\cos(2k+1)\pi x\,dx,\qquad b_{2k+1} = 2\int_0 f\sin(2k+1)\pi x\,dx,
$$

and the formal series for f_2 becomes

$$\sum_{k=0}^{\infty}\{a_{2k+1}\cos(2k+1)\pi x+b_{2k+1}\sin(2k+1)\pi x\}$$

which is identified at once as the formal expansion of $f(x)$ in terms of (10) on (0,1).

COROLLARY I. *The term-by-term difference of the formal series for $f(x)$ in terms of the Fourier set* (9) *and of the set* (10) *converges uniformly to zero on any closed interval interior to* (0,1).

Proof. This term-by-term difference is the Fourier series on (0,2) for f_1-f_2; since $f_1-f_2\equiv 0,\ 0\leq x\leq 1$, it follows that this Fourier series converges uniformly to zero on any closed interval interior to (0,1), by one of the most elementary properties of Fourier series.

COROLLARY II. *If $f(x)$ is continuous and of bounded variation in a neighborhood on the right of $x=0$ and is continuous and of bounded variation in a neighborhood on the left of $x=1$, the formal series for $f(x)$ in terms of* (10) *converges at $x=0$ to $\frac{1}{2}(f(0+)-f(1-))$ and at $x=1$ to $\frac{1}{2}(-f(0+)+f(1-))$.*

Proof. This is a direct consequence of the fact that the series is a Fourier series for $f_2(x)$ on (0,2). It is sufficient to recall that under the conditions stated this Fourier series converges at $x=0$ to $\frac{1}{2}(f_2(0+)+f_2(2-))$ and at $x=1$ to $\frac{1}{2}(f_2(1-)+f_2(1+))$.

It is clear that a large number of properties of Fourier series will be carried over by Theorem I and Corollary I to the series formed from (10).* In particular there exists no summable function other than a function identically zero almost everywhere on (0,1) such that its Fourier-like coefficients with respect to (10) all vanish. Again we see by Corollary I that all properties of divergence and of both uniform and non-uniform convergence and summability are shared alike on any closed interval interior to (0,1) by the expansions formed from the Fourier set (9) and the set (10).

---

* It is of interest to notice that analogous considerations for the Fourier series on $(0,2\pi)$ will lead to the series in terms of the sine set and of the cosine set on $(0,\pi)$,

$$\sqrt{\frac{2}{\pi}}\sin x,\ \sqrt{\frac{2}{\pi}}\sin 2x,\cdots;\qquad \sqrt{\frac{1}{\pi}},\ \sqrt{\frac{2}{\pi}}\cos x,\ \sqrt{\frac{2}{\pi}}\cos 2x,\cdots.$$

The term-by-term difference of the formal expansions of any summable function on $(0,\pi)$ in terms of these two sets converges uniformly to zero on any closed interval interior to $(0,\pi)$. For functions summable with summable square this was proved by Walsh and Wiener, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 1 (1922), pp. 103–122, especially pp. 115–116.

Gibbs' phenomenon is also common to the two series in the neighborhood of any interior point of the unit interval for which it occurs for the series from (9). The behavior of an expansion in terms of (10) on the entire unit interval may be discussed by means of the Fourier series for f_2 on (0,2). The set (10) is thus of considerable interest as a set of trigonometric functions distinct from the Fourier set but resembling it very closely and arising from a similar differential system. We may also observe that expansions in terms of (10) can be investigated separately from the Fourier series by means of an integral analogous to the Dirichlet integral which is central in the theory of the latter.

III. The transformation $t = P(x)$

It is evident at a glance that the transformation $t = P(x)$ is of fundamental importance in the discussion of the series (8.1) and (8.2) since, no matter what summable function is used in calculating the coefficients, any sum attributed to the series must be of the form $f[P(x)]$. The relation of this sum function, if it exists, to the function for which the formal series is written down will depend on the nature of this transformation and its application to the integrals which are the coefficients of the series. We therefore devote a section to a detailed study of the relation $t = P(x)$.

From the assumptions made in § I concerning the function $p(x)$ we may immediately deduce the following theorem which describes the manner of oscillation of the continuous function $P(x) = \int_0^x p\,dx$:

Theorem II. *The interval* (0,1) *can be divided into a finite set of non-overlapping intervals* $\xi_1, \xi_2, \cdots, \xi_m$ *such that*

(*a*) *the set covers the whole unit interval*;

(*b*) *on* ξ_i *the function* $P(x)$ *either always increases or always decreases and furthermore* $P(x)$ *lies between consecutive integers*: $k \leqq P(x) \leqq k+1$, $k = -M, -M+1, \cdots, -1, 0, 1, \cdots, +M$;

(*c*) *if on* ξ_i *the function* $P(x)$ *always increases* (*decreases*) *and lies between consecutive integers* k *and* $k+1$, *then on the two adjacent* ξ-*intervals* $P(x)$ *either always decreases* (*increases*) *or does not lie between the integers* k *and* $k+1$.

Proof. On any one of the intervals ϱ_i of § I, $p(x)$ is either positive or negative almost everywhere. For the purpose of the argument we may assume the first possibility to be the case. Then for any two points x_1, x_2 on that interval we have $P(x_2) - P(x_1) = \int_{x_1}^{x_2} p\,dx > 0$, $x_2 > x_1$, since there exists on (x_1, x_2) a measurable set of points with measure greater than zero on which $p(x)$ is positive. The same type of reasoning applies when $p(x)$ is negative on ϱ_i rather than positive.

Each interval ϱ_i, on which $P(x)$ is now known to be either always increasing or always decreasing, can be divided into a finite number of sub-intervals on each of which $P(x)$ lies between consecutive integers and on no two of which $P(x)$ lies between the same pair of integers. The number of sub-intervals thus determined is finite since $P(x)$ is continuous.

The sub-intervals just defined form a set of intervals $\xi_1, \xi_2, \cdots, \xi_m$ with all the properties set forth in the statement of the theorem.

It is now an exceedingly simple matter to determine the manner in which the intervals ξ_i are transformed by the relation $t = P(x)$. We have

THEOREM III. *The transformation $t = P(x)$ takes the set of intervals ξ_i into a set of intervals τ_i such that*

(*a*) *τ_i has positive or negative sense according as $P(x)$ increases or decreases on ξ_i;*

(*b*) *the algebraic sum of the intervals τ_i is the unit t-interval;*

(*c*) *τ_i lies between successive integers.*

Proof. The interval $\xi_i : (a_i, b_i)$ clearly goes over into the interval τ_i: $[P(a_i), P(b_i)]$. Statements (a) and (c) are then made apparent by Theorem II (b). It is also easy to see that if $0, x_1, x_2, \cdots, x_{n-1}, 1$ are the end points of the intervals ϱ_i of § I the algebraic sum of the intervals τ_i is found by adding the intervals $[P(0), P(x_1)]$, $[P(x_1), P(x_2)]$, $\cdots$, $[P(x_{n-1}), P(1)]$ algebraically. The sum is then seen to be the interval $[P(0), P(1)]$ which is precisely the interval (0,1).

Before going on to discuss the transformation of integrals, we shall state two necessary lemmas of a general nature. Of these we shall prove the

LEMMA I. If $t = t(x)$ *is the Lebesgue integral* $\int_a^x g(x)\,dx + a'$ *of a summable function* $g(x)$ *which is positive almost everywhere* $a \leqq x \leqq b$, *then*

(*a*) $t = t(x)$ *has a unique inverse* $x = x(t)$ *which is a one-valued always increasing function of* t, $a' \leqq t \leqq b'$;

(*b*) $x(t)$ *is equal to the Lebesgue integral* $\displaystyle\int_{a'}^{t} \frac{dt}{g[x(t)]} + a$.

If $t = t(x)$ *is the Lebesgue integral* $\int_a^x g(x)\,dx + a'$ *of a summable function* $g(x)$ *which is negative almost everywhere* $a \leqq x \leqq b$, *then*

(*a*) $t = t(x)$ *has a unique inverse* $x = x(t)$ *which is a one-valued always decreasing function of* t, $b' \leqq t \leqq a'$;

(*b*) $x(t)$ *is equal to the Lebesgue integral* $\displaystyle\int_{a'}^{t} \frac{dt}{g[x(t)]} + a$.

Proof. As we have already seen in the proof of Theorem II, $t(x)$ is an always increasing continuous function of x when $g(x)$ is positive almost everywhere on (a, b); statement (a) of the first part of the lemma follows

immediately from this fact. Turning now to (b) in the first part of the lemma, we examine the equality

$$\frac{x(t+h)-x(t)}{h} = \frac{h'}{t(x+h')-t(x)},$$

where h, h' vanish together. The reciprocal of the term on the right has the limit $g(x)$ almost everywhere; since $g(x)$ is positive almost everywhere the term itself has a finite limit almost everywhere. Corresponding to the set of zero measure on the x-axis for which the limit does not exist or is not equal to $g(x)$ there is a set of zero measure on the t-axis. To establish this statement we note that the measure of the set of points on the t-axis corresponding to a finite or denumerably infinite set of non-overlapping intervals E on the x-axis is equal to the integral $\int_E g dx$. We can now enclose any set of zero measure on the x-axis in an open point set of measure less than δ*, that is, in a finite or denumerably infinite set of non-overlapping closed intervals E†. By choosing δ sufficiently small we can make the measure of E', which corresponds to E and therefore contains the set corresponding to the set of zero measure on the x-axis, less than any preassigned positive ϵ:

$$m(E') = \int_E g(x)dx < \epsilon, \qquad m(E) < \delta.$$

Thus we must have the result that $\lim_{h\to 0}(x(t+h)-x(t))/h$ exists and is equal to $1/g[x(t)]$ almost everywhere, $a' \leqq t \leqq b'$.

If we can show that $x(t)$ is a Lebesgue integral we can then state that the function is the indefinite integral of its differential coefficient, $x = \int_{a'}^{t} \frac{dt}{g[x(t)]} + a$. It is well known that a necessary and sufficient condition that a function be an integral in the sense of Lebesgue is that it be absolutely continuous.‡ We shall prove that this is true for $x(t)$; that is, we shall show that $\sum |x(b_i')-x(a_i')|$ formed for any set of non-overlapping closed intervals E' can be made uniformly small by taking $m(E')$ sufficiently small. In other words, if we are given any positive ϵ, however small, we can find a positive δ such that if $m(E') < \delta$ then $m(E) < \epsilon$, where E is the set of intervals on the x-axis corresponding to E'. For the set of points on which $g \leqq \sqrt{\delta}$ is of measure $\eta(\delta)$ vanishing with δ; we pick $\sqrt{\delta} + \eta(\delta) < \epsilon$ and require $m(E') < \delta$.

* de la Vallée Poussin, loc. cit., § 21.

† de la Vallée Poussin, loc. cit., § 14.

‡ Vitali, Atti della Reale Accademia di Torino, vol. 40 (1905), pp. 1021–1034; Lebesgue, loc. cit., p. 129, footnote.

Splitting E into the measurable sets of points E_1, E_2 on which $g > \sqrt{\delta}$, $g \leqq \sqrt{\delta}$ respectively, we see that

$$\sqrt{\delta}\, m(E_1) \leqq \int_{E_1} g\, dx \leqq \int_E g\, dx = m(E') < \delta,$$

whence

$$m(E) = m(E_1) + m(E_2) \leqq \sqrt{\delta} + \eta(\delta) < \epsilon.$$

The proof of the first part of the lemma is thus completed. The second part, in which $g(x)$ is negative almost everywhere, is proved in the same manner.

The second lemma we state without proof. It is

LEMMA II.* *If $x = x(t)$ is the Lebesgue integral of a summable function $h(t)$ which is positive (negative) almost everywhere on (a', b'), then for any summable function $f(x)$*

$$\int_a^b f(x)\, dx = \int_{a'}^{b'} f[x(t)]\, \frac{dx(t)}{dt}\, dt = \int_{a'}^{b'} f[x(t)]\, h(t)\, dt,$$
$$b = x(b'), \qquad a = x(a').$$

With the aid of Lemmas I and II we are now able to demonstrate

THEOREM IV. *If $p(x)f(x)$ is summable, $0 \leqq x \leqq 1$, then*

$$\int_{\xi_i} pf\, dx = \int_{\tau_i} p[X_i(t)]\, f[X_i(t)]\, \frac{dX_i(t)}{dt}\, dt = \int_{\tau_i} f[X_i(t)]\, dt$$

where $X_i(t)$ is the inverse of $t = P(x)$ for the interval ξ_i; and, conversely, if $f(t)$ is summable and is defined outside the unit interval by the periodic relation $f(t+1) = f(t)$ or by the anti-periodic relation $f(t+1) = -f(t)$, then

$$\int_{\tau_i} f(t)\, dt = \int_{\xi_i} f[P(x)]\, \frac{dP(x)}{dx}\, dx = \int_{\xi_i} p(x)\, f[P(x)]\, dx.$$

Proof. On the interval ξ_i, the function $t = P(x)$ is the Lebesgue integral of a function positive (negative) almost everywhere on that interval, so that by Lemma I we can invert $t = P(x)$ by means of the function

$$X_i = \int_{a_i'}^{t} \frac{dt}{p[X_i(t)]} + a_i$$

---

* Lebesgue, Annales de la Faculté des Sciences de Toulouse, ser. 3, vol. 1 (1909), pp. 25–117, especially p. 44; Hobson, *The Theory of Functions of a Real Variable*, second edition, Cambridge University Press, 1921, § 440, pp. 592–595.

for t on $\tau_i:(a_i', b_i')$ and x on $\xi_i:(a_i, b_i)$. We then apply the second Lemma to each of the two integrals $\int_{\xi_i} pf\,dx$, $\int_{\tau_i} f\,dt$ remembering that $1/p[X_i(t)]$ is positive almost everywhere or negative almost everywhere on τ_i according as p is positive or negative almost everywhere on ξ_i. It should be noted that since p is bounded, pf is certainly summable if f is.

In order to pass to the consideration of integrals like those which appear as the coefficients in (8.1) and (8.2), we introduce new concepts. If we define $X_i(t)$ by the periodic relation $X_i(t+1) = X_i(t)$ for all the intervals $\tau_i + k$, $k = 0, \pm 1, \cdots$ then for some k, say k_i, one of these intervals will lie entirely on the unit interval and thus

$$\int_{\tau_i} f[X_i(t)]\,dt = \int_{\tau_i+k_i} f[X_i(t)]\,dt.$$

The new interval $\tau_i + k_i$ will still have definite positive or negative sense; to eliminate this circumstance, we let τ_i' be the interval $\tau_i + k_i$ taken in the positive sense. We can then write

$$\int_{\tau_i} f[X_i(t)]\,dt = \int_{\tau_i'} \operatorname{sgn} \tau_i f[X_i(t)]\,dt$$

where $\operatorname{sgn} \tau_i = +1$ if τ_i has positive sense and $\operatorname{sgn} \tau_i = -1$ if τ_i has negative sense. With a view to the discussion of the series (8.1) we now define a function

$$g_1(t) = f_{11}(t) + f_{12}(t) + \cdots + f_{1m}(t),$$

where $f_{1i} \equiv \operatorname{sgn} \tau_i f[X_i]$ on τ_i' and is identically zero elsewhere on $(0, 1)$. Similarly, with a view to the treatment of (8.2) we define a second function

$$g_2(t) = f_{21}(t) + f_{22}(t) + \cdots + f_{2m}(t),$$

where $f_{2i} \equiv (-)^{k_i} \operatorname{sgn} \tau_i f[X_i]$ on τ_i' and is identically zero elsewhere on $(0, 1)$. These two functions are so related to $f(x)$ that we shall call them the first and second P-average functions for $f(x)$, respectively. The introduction of these functions is prompted by considerations which will be more apparent later in our work.

We now prove two theorems on integrals which will serve as the basis for our analysis of the series (8.1) and (8.2). We begin with

THEOREM V. *If $p(x)f(x)$ is summable, then $g_1(t)$ and $g_2(t)$ are summable and*

$$\int_0^1 p(x)f(x)\,dx = \int_0^1 g_1(t)\,dt.$$

If $p(x)f^2(x)$ is also summable, then $g_1(t)$ and $g_2(t)$ are both summable with summable square. If $f(x)$ is bounded, then $g_1(t)$ and $g_2(t)$ are also bounded.

Proof. If $p(x)f(x)$ is summable, then, by Theorem IV,

$$\int_{\xi_i} pf\,dx = \int_0^1 f_{1i}(t)\,dt$$

and

$$\int_{\xi_i} pf\,dx = (-1)^{k_i}\int_0^1 f_{2i}\,dt$$

so that f_{1i} and f_{2i} are summable. Since $g_1(t)$ is a linear combination of the f_{1i} and $g_2(t)$ of the f_{2i}, both $g_1(t)$ and $g_2(t)$ are summable. The addition of the equalities

$$\int_{\xi_i} pf\,dx = \int_0^1 f_{1i}\,dt$$

gives us the fact that

$$\int_0^1 pf\,dx = \int_0^1 g_1(t)\,dt.$$

If pf^2 is summable also, then $\int_{\xi_i} pf^2\,dx$ exists and by Theorem IV and the definition of the functions f_{1i}, f_{2i} is equal to

$$\int_0^1 f_{1i}^2\,dt = \int_0^1 f_{2i}^2\,dt.$$

Thus f_{1i}^2 and f_{2i}^2 are summable so that this is also true of g_1 and g_2. We may note that if $\int_0^1 pf^2\,dx$ exists the inequality

$$\left|\int_0^1 pf\,dx\right| \leqq \int_0^1 |pf|\,dx$$
$$= \int_0^1 \sqrt{|p|}\;\sqrt{|p|}\;|f|\,dx \leqq \sqrt{\int_0^1 |p|\,dx\cdot\int_0^1 |p|\,f^2\,dx}$$

proves the existence of $\int_0^1 pf\,dx$.

Lastly, if f is bounded so are f_{1i} and f_{2i}. In consequence g_1 and g_2 are also bounded.

The other theorem serves as a sort of converse to the one we have just proved. Its statement follows:

THEOREM VI. *If $h(t)$ is summable and is defined outside the unit interval by the periodic relation $h(t+1) = h(t)$ or the anti-periodic relation $h(t+1) = -h(t)$, then $p(x)\,h[P(x)]$ is summable and*

$$\int_0^1 ph[P]\,dx = \int_0^1 h(t)\,dt.$$

If $h(t)$ is in addition of summable square, then $ph^2[P]$ is summable. If $h(t)$ is bounded, then so is $h[P(x)]$.

Proof. In any of the three cases we can write

$$\int_0^1 h(t)\,dt = \sum_1^m \int_{\tau_i} h(t)\,dt = \sum_1^m \int_{\xi_i} p(x)h[P(x)]\,dx = \int_0^1 p(x)h[P(x)]\,dx.$$

Thus $ph[P]$ is summable. In case we have $h^2(t)$ summable we may replace $h(t)$ by $h^2(t)$ in the equalities just written down, obtaining $\int_0^1 h^2(t)\,dt = \int_0^1 ph^2[P]dx$ and in this way proving that $ph^2[P]$ is summable. It is obvious that if $h(t)$ is bounded, $h[P]$ is also bounded.

With Theorems V and VI we are prepared to treat the two series (8.1) and (8.2). It seems in place, however, to say a word about the possibility of extending the results of this section to transformations associated with a somewhat more general class of functions p. To obtain analogues of Theorems II and III it is sufficient to require that it be possible to determine a denumerably infinite set of non-overlapping intervals $\varrho_1, \varrho_2, \varrho_3, \cdots$ covering the unit interval except for a set X of zero measure and such that on ϱ_i the function $p(x)$ is either positive or negative almost everywhere. The analysis necessary to establish a theorem parallel to Theorem III is somewhat more complicated than that which we have found sufficient for the type of function $p(x)$ we are considering. Theorem IV remains the same. By investigating some simple convergence questions we can set up the P-average functions for any function $f(x)$, but Theorems V and VI are no longer true, at least for the general class of functions $p(x)$ just defined. To bring out the essential points of the treatment which we have outlined, we shall take up a simple example.

We shall define $p(x)$ so as to be constant on each of a denumerably infinite set of intervals $\varrho_1, \varrho_2, \cdots$. It is well known that the infinite series

$1+2\sum_1^\infty(1/n^2)$ converges to a positive constant C. The interval ϱ_1 shall be of length $1/C$ and shall have $x=1$ as an end point. Then ϱ_2 shall abut on ϱ_1 and be of length $1/4C$, ϱ_3 shall abut on ϱ_2 and be of length $1/4C$. In general, ϱ_{i+1} shall abut on ϱ_i, and ϱ_{2k}, ϱ_{2k+1} shall each be of length $1/((k+1)^2C)$. Then $\sum_1^\infty m(\varrho_i)=1$. We shall denote the end points of the intervals $\varrho_1, \varrho_2, \cdots$ by $1=a_0>a_1>a_2\cdots$. Now we put

$$\left.\begin{array}{ll} p(x)\equiv +C, & a_{2k+1}>x>a_{2k+2} \\ p(x)\equiv -C, & a_{2k}>x>a_{2k+1} \\ p(a_k)\equiv 0 & \end{array}\right\} \quad k=0,1,2,\cdots.$$

A few simple computations show that $P(x)=\int_0^x p(x)\,dx$ is a linear function of x on each ϱ-interval, while $P(a_{2k})=1/(k+1)^2$, $k=0,1,2,\cdots$ and $P(a_{2k+1})=0$, $k=0,1,2,\cdots$. The ϱ-intervals of this example are precisely the ξ-intervals analogous to those of Theorem II. By the transformation $t=P(x)$ there corresponds to the interval ϱ_{2k+1} the interval $\tau_{2k+1}:(0,\,1/(k+1)^2)$ taken in the positive sense, and to the interval ϱ_{2k} the interval $\tau_{2k}:(0,\,1/(k+1)^2)$ taken in the negative sense. Since all the intervals τ_i are on the unit interval, the two P-average functions coincide in this case. In constructing this one we will obtain instead of a finite sum an infinite series which, however, has the peculiarity that on any interval $(\epsilon, 1)$, $\epsilon>0$, it is actually a finite sum. As an example we may take the bounded summable function which is identical to 1 on $\varrho_1, \varrho_3, \varrho_5, \cdots$, and identical to zero elsewhere. We have at once

$$\int_{\rho_{2k+1}} p\,dx=\int_{\tau_{2k+1}} dt.$$

Thus if we put $f_{2k+1}(t)\equiv 1$ on τ_{2k+1} and let it vanish identically elsewhere we find that the P-average functions for this particular $f(x)$ are given by the infinite series $g(t)=\sum_{k=0}^\infty f_{2k+1}(t)$. It is quickly seen that $g(t)\equiv k$ on $[1/(k+1)^2,\,1/k^2]$. Now $f(x)$, being bounded, is of summable square; but, on the other hand, we see that

$$\int_0^1 g^2\,dt=\sum_{k=1}^\infty k^2\left[\frac{1}{k^2}-\frac{1}{(k+1)^2}\right]=\sum_{k=1}^\infty\frac{2k+1}{(k+1)^2},$$

a divergent series. Thus $g(t)$ is not of summable square, and Theorem V is not true as a whole for this transformation. In the same way we can

show that, although $\frac{1}{2}t^{-\frac{1}{2}}$ is summable, $\frac{1}{2}p(x)P^{-\frac{1}{2}}(x)$ is not; for if $\frac{1}{2}pP^{-\frac{1}{2}}$ were summable its integral over the set of intervals $\varrho_1, \varrho_3, \varrho_5, \cdots$ would exist and would be given by $\sum_{k=0}^{\infty}\int_{\rho_{2k+1}} pP^{-\frac{1}{2}}\,dx$. This last expression turns out to be a divergent series since

$$\sum_{k=0}^{\infty}\int_{\rho_{2k+1}} pP^{-\frac{1}{2}}\,dx = \sum_{k=0}^{\infty}\int_{\tau_{2k+1}} \frac{1}{2}t^{-\frac{1}{2}}\,dt = \sum_{k=0}^{\infty}\left[t^{\frac{1}{2}}\right]_{t=0}^{t=\frac{1}{(k+1)^2}} = \sum_{k=0}^{\infty}\frac{1}{k+1}.$$

Thus Theorem VI is not true as a whole for this special case.

The example which we have just discussed makes it clear why we have not attempted to take $p(x)$ of more general character, and serves to indicate along what lines any generalization can be expected to take place. With this we may leave the developments of this section.

IV. The behavior of the formal expansions

Instead of discussing the series (8.1) and (8.2) alone we shall consider a wider class of series including these two as special cases. If we are given any normal orthogonal set $\varphi_{1k}(t)$, where $\varphi_{1k}(t+1) = \varphi_{1k}(t)$, we can apply Theorem VI to the summable function $\varphi_{1i}(t)\varphi_{1k}(t)$, whence

$$\int_0^1 \varphi_{1i}\varphi_{1k}\,dt = \int_0^1 p\,\varphi_{1i}[P]\,\varphi_{1k}[P]\,dx = \begin{Bmatrix} 0 & i \neq k \\ 1 & i = k \end{Bmatrix}.$$

The set (9) is a special case. Similarly, if we are given a set of normal orthogonal functions $\varphi_{2k}(t)$ where $\varphi_{2k}(t+1) = -\varphi_{2k}(t)$ we find

$$\int_0^1 p\,\varphi_{2i}[P]\,\varphi_{2k}[P]\,dx = \begin{Bmatrix} 0 & i \neq k \\ 1 & i = k \end{Bmatrix}.$$

The set (10) is a special example. It is natural to form from the sets

$$\begin{array}{llll} \varphi_{11}[P], & \varphi_{12}[P], & \varphi_{13}[P], & \cdots, \\ \varphi_{21}[P], & \varphi_{22}[P], & \varphi_{23}[P], & \cdots \end{array}$$

formal series $\sum_1^{\infty} a_{1k}\varphi_{1k}[P]$ and $\sum_1^{\infty} a_{2k}\varphi_{2k}[P]$ respectively for any function of the class for which $a_{1k} = \int_0^1 pf\varphi_{1k}[P]\,dx$ and $a_{2k} = \int_0^1 pf\varphi_{2k}[P]\,dx$

exist. If $\varphi_{1k}, \varphi_{2k}$ are bounded then this class of functions is at least the class of all functions for which $p(x)f(x)$ is summable. If $\varphi_{1k}, \varphi_{2k}$ are summable with summable square then this class is at least the class of all functions for which $p(x)f^2(x)$ is summable. By Theorem VI the Lebesgue integrals $\int_0^1 p\,\varphi_{1k}^2[P]\,dx$, $\int_0^1 p\,\varphi_{2k}^2[P]\,dx$ exist. Because of the existence of $\int_0^1 pf^2\,dx$ the inequalities

$$\left|\int_0^1 pf\varphi_{1k}[P]\,dx\right| \leqq \int_0^1 |pf\varphi_{1k}[P]|\,dx$$
$$= \int_0^1 \sqrt{|p|}\,|f|\,\sqrt{|p|}\,|\varphi_{1k}[P]|\,dx \leqq \sqrt{\int_0^1 |p|f^2\,dx\cdot\int_0^1 |p|\,\varphi_{1k}^2[P]\,dx}\,,$$

$$\left|\int_0^1 pf\varphi_{2k}[P]\,dx\right| \leqq \int_0^1 |pf\varphi_{2k}[P]|\,dx$$
$$= \int_0^1 \sqrt{|p|}\,|f|\,\sqrt{|p|}\,|\varphi_{2k}[P]|\,dx \leqq \sqrt{\int_0^1 |p|f^2\,dx\cdot\int_0^1 |p|\,\varphi_{2k}^2[P]\,dx}$$

demonstrate the summability of the measurable functions $pf\varphi_{1k}, pf\varphi_{2k}$.

We come next to

THEOREM VII. *The Fourier-like coefficients of $f(x)$ in terms of the set of functions $\varphi_{11}[P], \varphi_{12}[P], \cdots$ are the Fourier-like coefficients of the first P-average function for $f(x)$ in terms of the set $\varphi_{11}(t), \varphi_{12}(t), \cdots$; and the Fourier-like coefficients of $f(x)$ in terms of the set $\varphi_{21}[P], \varphi_{22}[P], \cdots$ are the Fourier-like coefficients of the second P-average function for $f(x)$ in terms of the set $\varphi_{21}(t), \varphi_{22}(t), \cdots$.*

Proof. If $G_{1k}(t)$ is the first P-average function for the product $f\varphi_{1k}[P]$ then, by Theorem V, $\int_0^1 G_{1k}\,dt = \int_0^1 pf\varphi_{1k}[P]\,dt$. Now $G_{1k}(t)$ when evaluated is precisely $g_1(t)\varphi_{1k}(t)$ where g_1 is the first P-average function for $f(x)$. Hence $a_{1k} = \int_0^1 pf\varphi_{1k}[P]\,dx = \int_0^1 g_1(t)\varphi_{1k}(t)\,dt$. In the same way, if $G_{2k}(t)$ is the first P-average function for the product $f\varphi_{2k}[P]$ and g_2 is the second P-average function for $f(x)$ then

$$G_{2k}(t) = g_2(t)\varphi_{2k}(t), \qquad a_{2k} = \int_0^1 pf\varphi_{2k}[P]\,dx = \int_0^1 g_2(t)\varphi_{2k}(t)\,dt.$$

COROLLARY I. *If $\int_0^1 pf^2\,dx$ exists, a series formed from $\sum_0^\infty a_{1k}\varphi_{1k}[P]$ by grouping its terms suitably will converge essentially uniformly to $g_1^*[P]$ where*

$g_1^*(t+1) = g_1^*(t)$; *and a series formed from* $\sum_1^\infty a_{2k}\varphi_{2k}[P]$ *by grouping its terms properly will converge essentially uniformly to* $g_2^*[P]$ *where* $g_2^*(t+1) = -g_2^*(t)$.

Proof. By the hypothesis that $\int_0^1 pf^2dx$ exists we have the fact that the two P-average functions for $f(x)$ are summable with summable square. Consider the first of these, $g_1(t)$. By the Riesz-Fischer theorem, in the form given to it by Weyl, there exists a series formed by grouping the terms of the series $\sum_1^\infty a_{1k}\varphi_{1k}(t)$ which will converge essentially uniformly to a function $g_1^*(t)$, where $g_1(t)-g_1^*(t)$ is a function all of whose coefficients with respect to $\varphi_{11}(t)$, $\varphi_{12}(t)$, $\cdots$ vanish.† By essentially uniform convergence we mean that, given any decreasing sequence of positive ϵ''s with limit zero, $\epsilon_1' > \epsilon_2' > \epsilon_3' > \cdots$, we can find on the unit interval a sequence of measurable point sets E_1', E_2', E_3', $\cdots$ each contained in the succeeding set, with $m(E_k') > 1-\epsilon_k'$, such that on each set of the sequence the series converges uniformly to its limit. On the unit x-interval we now have a sequence of point sets E_1, E_2, E_3, $\cdots$ where E_k is the set of points for which $t = P(x)$ takes on values congruent (modulo 1) to a point of E_k'. We see that on E_k the series formed by grouping the terms of $\sum_1^\infty a_{1k}\varphi_{1k}[P]$ converges uniformly to $g_1^*[P]$ because of the periodicity of the functions $\varphi_{1k}(t)$. Now CE_k, the set complementary to E_k, consists of all the points of the unit x-interval for which $t = P(x)$ is not congruent (modulo 1) to a point of E_k'; that is, of all the points for which $t = P(x)$ is congruent to a point of CE_k'. Thus CE_k can be obtained as the sum of the point sets e_{1k}, e_{2k}, e_{3k}, $\cdots$, e_{mk}, where e_{ik} is the set on ξ_i corresponding by the relation $t = P(x)$ to a set congruent (modulo 1) to CE_k'. Since $m(CE_k') < \epsilon_k'$ we can enclose CE_k' in an open set O_k' of measure $< 2\epsilon_k'$. On the interval ξ_i the set o_{ik} corresponding to O_k' contains e_{ik}. Since for values of t on τ_i we have

$$x = X_i(t) = \int_{\tau_i} \frac{dt}{p[X_i]} + a_i$$

it follows that $m(o_{ik})$ is given by the Lebesgue integral of $\operatorname{sgn}\tau_i/p[X_i]$ taken over the points of O_k' on τ_i'. From this point on we apply the arguments we used in discussing a similar situation in the proof of Lemma I. By choosing an ϵ_k' small enough we can make $m(CE_k) = \sum_{i=1}^m m(e_{ik}) \leqq \sum_{i=1}^m m(o_{ik})$ less than any preassigned positive ϵ. Thus, given a sequence of decreasing positive ϵ_k's with limit zero, we can find a subsequence of E_1, E_2, E_2, $\cdots$, say E_1'', E_2'', E_3'', $\cdots$, such that $m(E_k'') > 1-\epsilon_k$. The

† Plancherel, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1910), pp. 289–335, chap. I.

series formed by grouping the terms of $\sum_1^\infty a_{1k}\varphi_{1k}[P]$ therefore converges essentially uniformly to the function $g_1^*[P]$ where $g_1^*(t+1) = g_1^*(t)$. If the set $\varphi_{1k}(t)$ is closed with respect to all functions summable with summable square—that is, if the only function of this class which has all zero coefficients with respect to the set is identically zero except on a set of measure zero—then $g_1^*(t) = g_1(t)$ almost everywhere. Exactly the same sort of reasoning applies to the function $g_2(t)$ and the series $\sum_1^\infty a_{2k}\varphi_{2k}(t)$, the only difference being that here the functions $\varphi_{2k}(t)$ are anti-periodic instead of periodic.

COROLLARY II. *If the set $\varphi_{1k}(t)$ is closed with respect to all functions summable with summable square, and if $f_1(x)$ and $f_2(x)$ are two functions such that $\int_0^1 pf_1^2\,dx$ and $\int_0^1 pf_2^2\,dx$ exist, then a necessary and sufficient condition that $f_1(x)$ and $f_2(x)$ have the same coefficients with respect to $\varphi_{11}[P], \varphi_{12}[P], \cdots$ is that their first P-average functions coincide almost everywhere. If the set φ_{1k} is closed with respect to all summable functions, and if $f_1(x)$ and $f_2(x)$ are two functions such that $\int_0^1 pf_1\,dx$ and $\int_0^1 pf_2\,dx$ exist, then a necessary and sufficient condition that $f_1(x)$ and $f_2(x)$ have the same coefficients with respect to the set $\varphi_{11}[P], \varphi_{12}[P], \cdots$ is that their first P-average functions coincide almost everywhere. Similar statements involving the second P-average functions can be made for the set $\varphi_{21}[P]$, $\varphi_{22}[P], \cdots$.*

As we have already remarked in § II the sets (9) and (10) are closed with respect to all summable functions.

Corollary I to Theorem VII suggests a theorem which we shall prove independently as

THEOREM VIII. *If the function $P(x)$ is not monotone, there are infinitely many non-null functions whose P-average functions are both identically zero.*

Proof. By a non-null function we mean any function which is not a null function; and a null function is any function identically zero except on a set of measure zero. When we assume that $P(x)$ is not monotone we assume that there are two or more of the intervals ϱ_i described in § I. For the sake of definiteness we shall assume that on $\varrho_1:(0, a)$ the function $P(x)$ increases from zero to $P(a)$ while on $\varrho_2:(a, b)$ it decreases from $P(a)$ to $P(b)$. If $P(b) \geqq 0$ we consider the interval R_1 made up of ϱ_2 and the sub-interval of ϱ_1 for which $P(b) \leqq P(x) \leqq P(a)$; in case $P(b) \leqq 0$ we take the interval R_2 composed of ϱ_1 and the sub-interval of ϱ_2 for which $0 \leqq P(x) \leqq P(a)$. If there is given any bounded summable function $h(t)$ defined for all values of t on the interval $0 \leqq t \leqq P(a)$, we set $f(x)$ identically equal to $h[P]$ on R_1 or R_2 as the case may be, and identically equal to zero elsewhere, $0 \leqq x \leqq 1$. It is then not difficult

to see by reference to the definitions of the two P-average functions for $f(x)$ that both these functions vanish in the case of the bounded summable function which we have constructed. All the coefficients of $f(x)$ with respect to the sets $\varphi_{11}[P]$, $\varphi_{12}[P], \cdots$ and $\varphi_{21}[P]$, $\varphi_{22}[P], \cdots$ exist and vanish by Theorem VII. It is, of course, unnecessary but simpler to confine ourselves to bounded functions.

Theorem VII, Corollary I, and Theorem VIII reveal the striking peculiarities of the expansions of § I and their generalizations; peculiarities entirely new, so far as we are informed, in expansion problems. For any function $f(x)$ such that $\int_0^1 p(x)f^2(x)\,dx$ exists, the formal series will always represent in a certain sense a definite function, which will be of the specialized form $g_1^*[P]$ or $g_2^*[P]$. It is clear that if $\int_0^1 pf^2\,dx$ exists then $f(x)$ can be represented by its formal series in terms of a closed set $\varphi_{11}[P]$, $\varphi_{12}[P], \cdots$ if and only if its functional values are so distributed that it is of the form $g_1[P]$, $g_1(t+1) = g_1(t)$ except possibly on a set of measure zero; and by its formal series in terms of a closed set $\varphi_{21}[P]$, $\varphi_{22}[P], \cdots$ if and only if its functional values are so distributed that it is of the form $g_2[P]$, $g_2(t+1) = -g_2(t)$ except possibly on a set of measure zero. There are infinitely many functions all having the same formal expansions in terms of the two general sets of functions which we are discussing, and this quite apart from any considerations of the closure of the sets $\varphi_{11}(t)$, $\varphi_{12}(t), \cdots$ and $\varphi_{21}(t)$, $\varphi_{22}(t), \cdots$. The remarks which have just been made are, of course, expected to apply only in case $P(x)$ is not monotone. If $P(x)$ is monotone the expansions have no unusual properties.

The detailed consideration of a formal series here is thrown back on to that of the series $\sum_1^\infty a_{1k}\varphi_{1k}(t)$ for $g_1(t)$ or the series $\sum_1^\infty a_{2k}\varphi_{2k}(t)$ for $g_2(t)$. We do not intend to go deeply into any such discussion, but will close with one theorem bearing on the series (8.1) and (8.2):

Theorem IX. *If the function $f(x)$ is of bounded variation and if its first P-average function is extended by the periodic relation $g_1(t+1) = g_1(t)$ then the formal series* (8.1) *converges at every point x for which $P(x) = t_0$ to the value $\frac{1}{2}(g_1(t_0+0)+g_1(t_0-0))$; and if its second P-average function is extended by the relation $g_2(t+1) = -g_2(t)$ then the formal series* (8.2) *converges at every point x for which $P(x) = t_0$ to the value $\frac{1}{2}(g_2(t_0+0)+g_2(t_0-0))$.*

Proof. The function $f(x)$ can be represented as the difference of two monotone increasing functions f_1 and f_2 because it is of bounded variation. Now the functions $f_{1i}(t)$ and $f_{2i}(t)$ used in defining the P-average functions for $f(x)$ are constant multiples of $f[X_i(t)]$ on τ_i' and zero elsewhere. $X_i(t)$ is periodic of period 1 and is monotone on τ_i'. Since a monotone function

of a monotone function is monotone, both $f_1[X_i(t)]$ and $f_2[X_i(t)]$ are monotone on τ_i'; and $f[X_i] = f_1[X_i] - f_2[X_i]$ is therefore of bounded variation on τ_i'. Thus $g_1(t)$ and $g_2(t)$ are also of bounded variation, being linear combinations of functions of bounded variation. Now the series for g_1 in terms of the set (9) is the Fourier series for a function of bounded variation and therefore converges to $\frac{1}{2}(g_1(t_0+0)+g_1(t_0-0))$ at t_0. The substitution of $P(x)$ for t gives the first part of the theorem as stated. Likewise, as we saw in Theorem I, the series for $g_2(t)$, where $g_2(t+1) = -g_2(t)$, in terms of (10) is the Fourier series on the interval $(0, 2)$ for a function of bounded variation. The series converges at the point t_0 to the value $\frac{1}{2}(g_2(t_0+0)+g_2(t_0-0))$. The substitution of $P(x)$ for t completes the theorem.

Other theorems of a similar nature will readily suggest themselves to the reader. We shall not go into any further detail, having obtained by the work up to this point a fairly complete idea of the character of the expansion problems arising from the two differential systems (1) and (2).

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ON THE INDEPENDENCE OF PRINCIPAL MINORS OF DETERMINANTS*

BY

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Introduction

A principal minor, or coaxial minor, of a determinant A is a minor obtained by striking out from A the same rows as columns. There are $2^n - 1$ principal minors of a determinant of the nth order, the determinant itself being included, but only $n^2 - n + 1$ of them are independent.†

For $n = 1, 2, 3$ the principal minors are all independent and for $n = 4$ the relations between them have been quite extensively studied.‡ For $n > 4$ little has been published either as to which minors constitute an independent set or as to the relations between the minors.

It is one of the purposes of this paper to determine several different types of complete sets of independent principal minors of the general determinant of the nth order and to show how the elements of the determinant may be expressed in terms of the minors of an independent set.

If we have a second determinant of the nth order with elements independent of the elements of the first determinant, there is a definite set of determinants obtained by replacing one or more columns of the first determinant by the corresponding column or columns of the second determinant. The set of principal minors of all such determinants is greatly enlarged over the set from the single determinant. It is the second purpose of this paper to determine several types of complete sets of independent principal minors of this enlarged set. There is also determined in this paper a complete set of independent principal minors of the determinants obtained when the above process is extended by adjoining to the original determinant more than one determinant of the nth order with independent elements.

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† See MacMahon, *Philosophical Transactions of the Royal Society of London*, vol. 185, pp. 111–160; Muir, *Philosophical Magazine and Journal of Science*, ser. 5, vol. 38, pp. 537–541.

‡ See, e.g., MacMahon, loc. cit.; Nansen, *Philosophical Magazine and Journal of Science*, ser. 5, vol. 44, pp. 362–67; Muir, *Transactions of the Royal Society of Edinburgh*, vol. 39, pp. 323–339.

It is the third purpose of this paper to prove the independence of certain sums of principal minors. In the fourth section of the paper the independence of these sums is used to determine the possibility of expressing polynomials as determinants with linear elements.

I

Let A represent a determinant of the nth order in which the element in row i and column j is denoted by a_{ij}. Furthermore, let $(A_{klmn\cdots})$ denote the principal minor obtained by striking out in A all columns and rows except those numbered $k, l, m, n, \cdots$. Thus,*

$$(A_i) = a_{ii}, \qquad (A_{ij}) = \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \cdots.$$

It has been shown by Muir† that the values of the principal minors of a determinant A with general elements a_{ij} are not affected if certain sets of $n-1$ of the elements are put equal to unity. For instance, if we divide the elements of the second column by a_{12}, of the third column by a_{13}, of the fourth column by a_{14}, and so forth, and also multiply the elements of the second row by a_{12}, of the third row by a_{13}, of the fourth row by a_{14}, and so forth, there is no effect either upon the values of the principal minors or upon the independence of the individual elements, except that $a_{1j} = 1\,(j = 2, 3, \cdots, n)$. The fact that there are only n^2-n+1 independent elements remaining is Muir's proof that there cannot be more than n^2-n+1 independent principal minors.

It may be shown similarly that there is no loss of generality in assuming $a_{ij} = 1 \ (i+1=j)$ in place of $a_{1j} = 1$.

THEOREM I. *The minors*

$$(1) \qquad (A_i), \ (A_{ij}), \ (A_{1ij}) \quad (i, j = 1, 2, \cdots, n;\ i<j)$$

constitute a total of n^2-n+1 *independent principal minors.*

We shall prove this theorem by showing that the functional matrix of these minors with respect to the elements a_{ij} contains a non-vanishing determinant of order n^2-n+1. In order to simplify the process we shall assume that $a_{1j} = 1\ (j = 2, 3, \cdots, n)$ and also that, after the partial derivatives with respect to the remaining a_{ij} have been formed, the elements $a_{ij}\ (i>j>1)$ are all put equal to zero. Then all $\partial(A_i)/\partial a_{ml} = 0$ except

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* We shall assume throughout this paper that the subscripts in the symbols for principal minors have been arranged in order of ascending magnitude.

† Muir, Philosophical Magazine and Journal of Science, ser. 5, vol. 38, pp. 537-541.

that $\partial(A_i)/\partial a_{ii} = 1$; all $\partial(A_{ij})/\partial a_{ml} = 0$ $(1 < m, m \neq l)$ except that $\partial(A_{ij})/\partial a_{ji} = -a_{ij}$; all $\partial(A_{1ij})/\partial a_{ml} = 0\,(1 < m < l)$ except that $\partial(A_{1ij})/\partial a_{ij} = a_{j1}$. Under these conditions there is in the functional matrix a determinant of order $n^2 - n + 1$ which may be expressed as the product of the three determinants

$$\frac{\partial\,[(A_1), (A_2), \cdots, (A_n)]}{\partial\,[\,a_{11},\ a_{22},\ \cdots,\ a_{nn}\,]}, \tag{2}$$

$$\frac{\partial\,[(A_{12}), (A_{13}), \cdots, (A_{1n}), (A_{23}), (A_{24}), \cdots, (A_{2n}), \cdots, (A_{n-1,n})]}{\partial\,[\,a_{21},\ a_{31},\ \cdots,\ a_{n1},\ a_{32},\ a_{42},\ \cdots,\ a_{n2},\ \cdots,\ a_{n,n-1}\,]}, \tag{3}$$

$$\frac{\partial\,[(A_{123}), (A_{124}), \cdots, (A_{12n}), (A_{134}), (A_{135}), \cdots, (A_{13n}), \cdots, (A_{1,n-1,n})]}{\partial\,[\,a_{23},\ a_{24},\ \cdots,\ a_{2n},\ a_{34},\ a_{35},\ \cdots,\ a_{3n},\ \cdots,\ a_{n-1,n}\,]}. \tag{4}$$

These determinants are all different from zero since in each of them the elements of the principal diagonal are different from zero and all other elements are equal to zero. The minors (1) are therefore independent.

The explicit expressions for A and for the principal minors of A in terms of the minors (1) are obtained by expressing the elements of A in terms of the minors (1). Under the condition that $a_{1j} = 1$ $(j > 1)$, we have

$$\begin{aligned} (A_i) &= a_{ii}, \\ (A_{1j}) &= a_{11}\, a_{jj} - a_{j1}, \\ (A_{ij}) &= a_{ii}\, a_{jj} - a_{ij}\, a_{ji} \quad (i > 1), \\ (A_{1ij}) &= a_{11}\, A_{ij} - a_{i1}\, a_{jj} + a_{j1}\, a_{ij} + a_{i1}\, a_{ji} - a_{j1}\, a_{ii}, \end{aligned} \tag{5}$$

whence

$$\begin{aligned} a_{ii} &= (A_i), \\ a_{j1} &= (A_1)(A_j) - (A_{1j}), \\ a_{ij}\, a_{ji} &= (A_i)(A_j) - (A_{ij}) \quad (i > 1), \\ a_{j1}\, a_{ij} + a_{i1}\, a_{ji} &= (A_{1ij}) - (A_1)(A_{ij}) + 2(A_1)(A_i)(A_j) - (A_j)(A_{1i}) - (A_i)(A_{1j}). \end{aligned} \tag{6}$$

The expressions for all the elements of A in terms of (A_i), (A_{ij}), (A_{1ij}) follow immediately from these equations.*

Another complete set of independent principal minors of A is obtained if the minors (A_{1ij}) of the set (1) are replaced by the minors (A_{12j}), (A_{123j}),

* MacMahon, (Philosophical Transactions of the Royal Society of London, vol. 185, p. 147) states that it is impossible to express a determinant of odd order in terms of its principal minors. By the above process this statement is shown to be incorrect.

$(A_{1234j}), \cdots, (A_{123\cdots n})$, where j in each case takes all the values up to n which are greater than the subscript immediately preceding it. The functional matrix will again be used in proving the independence of this set of n^2-n+1 principal minors. If we assume as before that $a_{1j}=1$ $(j=2,3,\cdots,n)$ and that a_{ij} $(i>j>1)$ are put equal to zero after the partial derivatives are formed, the matrix contains a determinant of order n^2-n+1 which consists of the product of the determinants (2) and (3) and the determinant

$$(7)\quad \frac{\partial\,[(A_{123}),(A_{124}),\cdots,(A_{12n}),(A_{1234}),(A_{1235}),\cdots,(A_{123n}),\cdots,(A_{123\cdots n})]}{\partial\,[\;a_{23},\quad a_{24},\quad\cdots,\quad a_{2n},\quad a_{34},\quad a_{35},\quad\cdots,\quad a_{3n},\quad\cdots,\quad a_{n-1,n}\;]}.$$

Under the above conditions

$$\frac{\partial\,(A_{12\cdots ij})}{\partial\,a_{ij}} = \pm a_{j1}\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_{22} & a_{23} & \cdots & a_{2,i-1} & a_{2i} \\ 0 & a_{33} & \cdots & a_{3,i-1} & a_{3i} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & a_{i-1,i-1} & a_{i-1,i} \end{vmatrix} \neq 0 \qquad (1<i<j)$$

and

$$\frac{\partial\,(A_{12\cdots ij})}{\partial\,a_{ml}} = 0 \qquad (m<l;\ m>i \text{ or } l>j).$$

Consequently, in the determinant (7) all elements above the principal diagonal are equal to zero and all elements of the principal diagonal are different from zero. We thus have

THEOREM II. *The minors*

$$(8)\qquad (A_i),\quad (A_{ij}),\quad (A_{12j}),\quad (A_{123j}),\quad \cdots,\quad (A_{123\cdots n})$$

form a complete set of independent principal minors of the general determinant A.

It is easily verified that the process of obtaining the expressions for the elements of A in terms of this second set of principal minors involves merely the successive solution of sets of two equations, one of which is linear and the other quadratic. However, the pairs of equations to be solved are not so easily written as before, since each pair has a different form from those which precede it.

Still another set of independent principal minors is obtained if the $\frac{1}{2}(n-1)(n-2)$ minors (A_{1ij}) of the set (1) are replaced by the minors $(A_{123}), (A_{234}), (A_{345}), \cdots, (A_{1234}), (A_{2345}), \cdots, (A_{123\cdots n})$. In order to

prove them independent, let us assume that $a_{ij} = 1\ (i+1=j)$, instead of $a_{1j} = 1$, and that $a_{ij}(i>j+1)$ are put equal to zero after the partial derivatives in the functional matrix are formed. Under these conditions the matrix contains a determinant of order $n^2 - n + 1$ which consists of the product of two non-vanishing determinants of the same form as (2) and (3) and the determinant

$$(9)\quad \frac{\partial[(A_{123}),\ (A_{234}),\ \cdots,\ (A_{n-2,n-1,n}),\ (A_{1234}),\ (A_{2345}),\ \cdots,\ (A_{123\cdots n})]}{\partial[\ a_{13},\quad a_{24},\quad \cdots,\quad a_{n-2,n},\quad a_{14},\quad a_{25},\quad \cdots,\quad a_{1n}\ ]}.$$

The latter determinant obviously has all elements of the principal diagonal different from zero and all elements above the principal diagonal equal to zero. We thus have

THEOREM III. *The minors*

$$(10)\quad (A_i),\quad (A_{ij}),\quad (A_{i,i+1,i+2,\cdots,i+r})\qquad (r=2,3,\cdots,n-1;\ i+r\leqq n)$$

form a complete set of independent principal minors of the general determinant A.

In order to express the elements of A in terms of the minors (10) it is again necessary to solve rather complicated pairs of equations, each pair consisting of a linear and a quadratic equation.

II

Let B denote a second determinant of the nth order with elements b_{ij}. A series of determinants of the nth order may be obtained by replacing one or more columns of A by the corresponding column or columns of B. Each of these new determinants gives a set of $2^n - 1$ principal minors.

Let us indicate by $(A_{ijk\cdots}\ B_{rst\cdots})$ the principal minor which belongs to the above set and which contains elements from columns $i, j, k, \cdots (i<j<k\cdots)$ of the determinant A and from columns $r, s, t, \cdots (r<s<t\cdots)$ of the determinant B. It is evident that the numbers $i, j, k, \cdots, r, s, t, \cdots$ must all be different.

Just as in the previous case, we can make $n-1$ of the elements, say $a_{1j}(j=2,3,\cdots,n)$ or $a_{ij}(i+1=j)$, equal to unity without affecting the values of the principal minors. It follows that there are not more than $2n^2 - n + 1$ independent principal minors.

THEOREM IV. *The minors*

$$(11)\quad (A_i),\ (B_i),\ (A_{1i}),\ (A_1\, B_i),\ (B_{1i}),\ (A_{1ij}),\ (A_{1i}\, B_j),\ (A_1\, B_{ij}),\ (B_{1ij})$$

form a complete set of independent principal minors of the determinants obtained by combining A and B.

In order to prove the set (11) independent let us first observe that an element a_{lm}, b_{lm} $(l \neq m)$ can occur only in minors having a subscript equal to l and another subscript equal to m. It follows that the functional matrix of the set (11) contains a determinant of order $2n^2 - n + 1$ consisting of the product of the determinants

$$(12) \qquad \frac{\partial[(A_1),\ (A_2),\ \cdots,\ (A_n)]}{\partial[\ a_{11},\quad a_{22},\ \cdots,\quad a_{nn}]}, \qquad \frac{\partial[(B_1),\ (B_2),\ \cdots,\ (B_n)]}{\partial[\ b_{11},\quad b_{22},\ \cdots,\quad b_{nn}]},$$

and of determinants of the forms

$$(13) \quad \frac{\partial[(A_{1i}),(A_1B_i),(B_{1i})]}{\partial[\ a_{i1},\quad b_{1i},\quad b_{i1}\ ]}, \quad \frac{\partial[(A_{1ij}),(A_{1i}B_j),(A_1\,B_{ij}),(B_{1ij})]}{\partial[\ a_{ij},\quad a_{ji},\quad b_{ij},\quad b_{ji}\ ]} \quad (1<i<j).$$

The determinants (12) are already known to be different from zero. If, after forming the determinants (13), we put $a_{ij} = 0\ (i > j > 1)$ and $b_{ij} = 0\ (i > j)$, it is immediately evident that each of them is equal to the product of the elements in its principal diagonal, all of which are different from zero.

By analogous processes we may prove

THEOREM V. *The set of principal minors*

$$(14) \quad \begin{gathered} (A_j),\quad (B_j),\quad (A_{1j}),\quad (A_1\,B_j),\quad (B_{1j}), \\ (A_{1,2,3,\cdots,i-1,i,j}),\ (A_{1,2,3,\cdots,i-1,i}\,B_j),\ (A_{1,2,3,\cdots,i-1}B_{ij}),\ (A_{2,3,\cdots,i-1}\,B_{1ij}) \\ (i \geqq 2), \end{gathered}$$

and the set

$$(15) \quad \begin{gathered} (A_j),\quad (B_j),\quad (A_{j,j+1}),\quad (A_j\,B_{j+1}),\quad (B_{j,j+1}),\quad (A_{i,i+1,\cdots,i+r}), \\ (A_{i,i+1,\cdots,i+r-1}B_{i+r}),\ (A_{i,i+1,\cdots,i+r-2}B_{i+r-1,i+r}),\ (A_{i+1,\cdots,i+r-2}\,B_{i,i+r-1,i+r}) \\ (r = 2, 3, \cdots, n-1;\ i + r \leqq n) \end{gathered}$$

each form a complete set of independent principal minors of the determinants obtained by combining A and B.

It should be noted that the sets (11), (14) and (15) are obtained from the sets (1), (8) and (10), respectively, by adjoining to each set the minors obtained by replacing in the determinants of the set the elements of the last column, the elements of the last two columns, and the elements of the first column and the last two columns, by the corresponding elements

b_{ij}, except that only a part of the minors of the second order thus obtained are used.

The n^2 minors

$$(B_i), \quad (A_i B_j), \quad (B_{ij}) \qquad (i, j = 1, 2, \cdots, n;\ i<j) \tag{16}$$

when arranged in the above order each contain an element b_{ij} not in the minors which precede it and not in the minors of A. We thus have

THEOREM VI. *The minors* (16) *together with any complete and independent set of principal minors of A form a complete set of independent principal minors of the determinants formed by combining A and B.*

The process of expressing the elements a_{ij} and b_{ij} in terms of the minors of any one of the above independent sets involves the simultaneous solution of linear and quadratic equations. The process can be carried out quite easily for the set composed of the minors (16) and (A_i), (A_{ij}), (A_{1ij}).

The above process of forming determinants by combining two determinants A and B may be extended by combining k determinants A, B, C, $D, \cdots$ of the nth order. Select for the first column of the new determinant the first column of one of the k determinants, for the second column the second column of one of the k determinants, and so forth. The principal minors of all such determinants of the nth order will now be considered for the purpose of determining a complete and independent set.

The symbol $(A_{ijk\ldots} B_{rst\ldots} C_{uvw\ldots})$ will be used to indicate the principal minor containing elements from columns $i, j, k, \cdots$ of A, from columns $r, s, t, \cdots$ of B, from columns $u, v, w, \cdots$ of C, and so forth.

Since $n-1$ of the kn^2 elements may again be assumed equal to unity, there cannot be more than kn^2-n+1 independent principal minors obtained from the whole set of combined determinants.

Among the minors of the set (11) there is a group in which elements of B appear in the last column only and another group in which elements of B appear in the last two columns only. If in each of the minors of the first group the elements of the last column are replaced by the corresponding elements of $C, D, \cdots$, successively, and if in each of the minors of the second group the elements of the next to the last column are replaced by the corresponding elements of $C, D, \cdots$, successively, we have, for each determinant $C, D, \cdots$, n^2 new principal minors. That each minor thus found is independent of the others and of the minors of the set (11) is evident from the fact that, with the proper arrangement, each of them contains an element c_{ij}, $d_{ij}, \cdots$ not in any of the preceding minors. Moreover, since the number of independent principal minors thus obtained is kn^2-n+1, the set is complete.

By exactly the same process we can obtain complete sets of minors for the enlarged system from either (14) or (16). The set (15) yields a complete set for $A, B, C, D, \cdots$ if the elements of the last column of the minors which contain a single column of elements from B and the elements of the first column of the minors which contain two columns of elements from B are replaced by the corresponding elements of $C, D, \cdots$, successively.

The minors to be added to any one of the sets (11), (14), (15), (16) may take many other forms, but the above systems are especially simple because the elements of $C, D, \cdots$ appear only linearly. The fact that all elements of $A, B, C, D, \cdots$ can be expressed in terms of the principal minors follows immediately from the fact that the elements of $C, D, \cdots$ appear linearly and from the fact that we have been able previously to express the elements of A and B in terms of principal minors.

III

Let us denote by I_α the sum of all the principal minors of order α of the determinant A of order n. That the n sums thus obtained are independent becomes evident if we put $a_{ij} = 0$ $(i \neq j)$ and observe that the sums I_α $(\alpha = 1, 2, \cdots, n)$ are then simply the n elementary symmetric functions of a_{ii} $(i = 1, 2, \cdots, n)$.

In the previous section we obtained new determinants of the nth order by combining the determinants $A, B, C, D, \cdots$. We shall now study the sums $I_{\alpha\beta\gamma\cdots}$ of the principal minors of these new determinants, where $I_{\alpha\beta\gamma\cdots}$ denotes the sum of all such principal minors containing α columns from A, β columns from B, γ columns from C, and so forth.

We shall first find a superior limit to the number of independent sums $I_{\alpha\beta\gamma\cdots}$ obtained by the combination of two or more determinants. For this purpose let us observe that $I_{\alpha\beta\gamma\cdots}$ may be obtained from A by successive applications of two different types of differential operators. The first type is of the form

$$D_a = \sum_{i=1}^{n} \frac{\partial}{\partial a_{ii}}.$$

If the kth successive application is denoted by $k!\,D_a^k$, we have

$$D_a A = I_{n-1}, \qquad D_a^2 A = I_{n-2}, \qquad \cdots, \qquad D_a^k A = I_{n-k}.$$

The second type is of the form

$$D_{ab} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \frac{\partial}{\partial a_{ij}}.$$

If the lth successive application is denoted by $l!\,D_{ab}^{l}$, we have

$$D_{ab}\,I_{n-k} = I_{n-k-1,1}, \quad D_{ab}^{2}\,I_{n-k} = I_{n-k-2,2}, \quad \cdots, \quad D_{ab}^{l}\,I_{n-k} = I_{n-k-l,l}.$$

The operator

$$D_{bc} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\,\frac{\partial}{\partial b_{ij}},$$

which is of the second type, applied successively gives

$$D_{bc}\,I_{n-k-l,l} = I_{n-k-l,l-1,1}, \quad \cdots, \quad D_{bc}^{m}\,I_{n-k-l,l} = I_{n-k-l,l-m,m}.$$

Thus the sum $I_{\alpha\beta\gamma\cdots}$ may be obtained from A by first operating with D_a successively $n-(\alpha+\beta+\gamma+\cdots)$ times, by then operating with D_{ab} successively $\beta+\gamma+\cdots$ times, by then operating with D_{bc} successively $\gamma+\cdots$ times, and so forth.

We shall next show that all $I_{\alpha\beta\gamma\cdots}$ are solutions of the system of differential equations

$$(17)\quad R_{rs}(f) \equiv \sum_{k=1}^{n}\left(a_{kr}\frac{\partial f}{\partial a_{ks}} - a_{sk}\frac{\partial f}{\partial a_{rk}} + b_{kr}\frac{\partial f}{\partial b_{ks}} - b_{sk}\frac{\partial f}{\partial b_{rk}} + \cdots\right) = 0$$
$$(r, s = 1, 2, \cdots, n).$$

The determinant $I_n = A$ is evidently a solution of every equation of (17). It is easy to see that the relation

$$R_{rs}(D_a A) - D_a(R_{rs} A) = -\left(\frac{\partial A}{\partial a_{rs}} - \frac{\partial A}{\partial a_{rs}}\right) = 0$$

is true, since terms containing the second derivatives vanish. But $R_{rs}(A)=0$, whence it follows that $R_{rs}(D_a A) = R_{rs}(I_{n-1}) = 0$. Therefore, I_{n-1} is a solution of the system (17) and by induction it follows at once that all I_α are solutions of (17).

Let us repeat the above process with the operator D_{ab} instead of D_a. We have

$$R_{rs}(D_{ab} I_\alpha) - D_{ab}(R_{rs} I_\alpha) = \sum_{k=1}^{n}\left[\left(b_{kr}\frac{\partial I_\alpha}{\partial a_{ks}} - b_{sk}\frac{\partial I_\alpha}{\partial a_{rk}}\right)\right.$$
$$\left. - \left(b_{kr}\frac{\partial I_\alpha}{\partial a_{ks}} - b_{sk}\frac{\partial I_\alpha}{\partial a_{rk}}\right)\right] = 0.$$

Since $R_{rs}(I_\alpha) = 0$, we have $R_{rs}(D_{ab} I_\alpha) = 0$. It follows that $I_{\alpha 1}$ is a solution of (17) and by induction it is again easily shown that all $I_{\alpha\beta}$ are solutions.

A continuation of the above process shows that all $I_{\alpha\beta\gamma\cdots}$ are solutions of (17).

The one relation $\sum_{r=1}^{n} R_{rr}(f) = 0$ between the equations (17) is obvious. That the remaining n^2-1 equations are independent for two or more sets of elements $a_{ij}, b_{ij}, c_{ij}, \cdots$ is easily seen by writing out the matrix of the equation and putting $a_{ik} = 0$ $(i \neq k)$ in this matrix.

The system (17) thus involves kn^2 variables, where k is the number of the determinants $A, B, C, \cdots$, and contains n^2-1 independent equations for $k > 1$. It follows that for $k > 1$ there cannot be more than $kn^2-(n^2-1)$ independent sums $I_{\alpha\beta\gamma\cdots}$.

It is interesting to observe that for $k = 2$ and $n > 2$ the number $\frac{1}{2}(n^2+3n)$ of sums $I_{\alpha\beta}$ is less than the superior limit n^2+1 fixed by the system of differential equations. As a matter of fact, there are $\frac{1}{2}(n^2-3n+2)$ other independent solutions of equations (17) in this case.*

We shall now prove

THEOREM VII. *The sums*

$$I_{\alpha\beta\gamma} \qquad (\alpha, \beta = 0, 1, 2, \cdots, n;\ \gamma = 0, 1, 2, 3) \tag{18}$$

form a complete and independent system for $k = 3$.

It is evident that there are n sums of the type I_α and $\frac{1}{2}n(n+1)$ of the type $I_{\alpha\beta}$ $(\beta = 1, 2, \cdots, n)$. Consequently, there are $\frac{1}{2}n(n+1)$ of the type $I_{\alpha\beta 1}$, $\frac{1}{2}n(n+1)-n$ of the type $I_{\alpha\beta 2}$, $\frac{1}{2}n(n+1)-(2n-1)$ of the type $I_{\alpha\beta 3}$. These minors make a total of $2n^2+1$.

Let us now put $a_{ij} = 0$ $(i \neq j)$ and investigate the resulting functional matrix of the sums (18) with respect to a_{ii}, b_{ij}, c_{ij}. There exists in this matrix a determinant of maximum order which consists of the product of the determinants

$$\frac{\partial[I_1, I_2, \cdots, I_n]}{\partial[a_{11}, a_{22}, \cdots, a_{nn}]}, \quad \frac{\partial[I_{01}, I_{11}, \cdots, I_{n-1,1}]}{\partial[b_{11}, b_{22}, \cdots, b_{nn}]}, \quad \frac{\partial[I_{001}, I_{101}, \cdots, I_{n-1,0,1}]}{\partial[c_{11}, c_{22}, \cdots, c_{nn}]}, \tag{19}$$

and of a determinant F consisting of the functional determinant of all the remaining sums of (18) with respect to all the elements b_{ij}, c_{ij} $(i \neq j)$ except $b_{i,i+1}$ $(i = 1, 2, \cdots, n-1)$. The three determinants (19) are different from zero since each is equal to the functional determinant of the n symmetric functions of $a_{11}, a_{22}, \cdots, a_{nn}$.

We shall prove that F is different from zero by showing that the coefficient of the highest power of a_{nn} which occurs in F is different from

* Cf. Stouffer, Proceedings of the London Mathematical Society, ser. 2, vol. 15, p. 222.

zero. This coefficient is the product of the determinant which corresponds to F when the order of A is $n-1$ and the functional determinant

$$(20)\quad \frac{\partial[I_{02},\ I_{011},\ I_{002},\ I_{03},\ I_{021},\ I_{012},\ I_{003},\ \cdots,\ I_{0n},\ I_{0,n-1,1},\ I_{0,n-2,2},\ I_{0,n-3,3}]}{\partial[b_{n,n-1},\ c_{n,n-1},\ c_{n-1,n},\ b_{n,n-2},\ c_{n,n-2},\ b_{n-2,n},\ c_{n-2,n},\ \cdots,\ b_{n1},\ c_{n1},\ b_{1n},\ c_{1n}]}.$$

After forming the partial derivatives in (20) let us put all $b_{ik}(i \neq k)$ and $c_{ik}\,(i \neq k)$ equal to zero except $b_{i,i+1}\,(i=1,2,\cdots,n-1)$, $b_{i+1,i}\,(i=1,2,\cdots,n-3)$, and $c_{i+1,i}\,(i=1,2,\cdots,n-1)$. A partial derivative of $I_{0\beta\gamma}$ with respect to b_{nr}, b_{rn}, c_{nr} or c_{rn} is then zero unless at least one of the determinants in $I_{0\beta\gamma}$ contains elements from the rth row and column and also from all rows and columns of higher number. It follows that (20) becomes the product of the determinants

$$(21)\quad \frac{\partial[I_{0,n-r+1},\ I_{0,n-r,1},\ I_{0,n-r-1,2},\ I_{0,n-r-2,3}]}{\partial[\ b_{nr},\ \ c_{nr},\ \ b_{rn},\ \ c_{rn}\ ]} \qquad (r=1,2,\cdots,n-2)$$

and the determinant

$$(22)\quad \frac{\partial[\ I_{02},\ \ I_{011},\ \ I_{002}\ ]}{\partial[b_{n,n-1},\ c_{n,n-1},\ c_{n-1,n}]}.$$

Each of the determinants (21) and (22) is different from zero since, under the above conditions upon b_{ij} and c_{ij}, each is the product of the elements of its principal diagonal, all of which are different from zero. Thus F does not vanish when the order of A is n, provided it does not vanish when the order of A is $n-1$. But it is easily verified that F is different from zero for $n=3$. Hence, by induction the sums (18) are proved to be independent. Since the set (18) contains the maximum number of independent sums, it is a complete system.

The nature of the determinants (19) and the fact that (20) reduces to the product of the elements of its principal diagonal prove the independence of the sums (18) and

$$(23)\qquad I_{\alpha\beta 0\delta\varepsilon\cdots},\qquad I_{\alpha 01\delta\varepsilon\cdots},\qquad I_{\alpha\beta 2\delta\varepsilon\cdots}\qquad (\delta+\varepsilon+\cdots=1),$$

for $k=4,5,\cdots$, and any n, provided that they are independent for $n=3$. This latter fact is easily verified.

Since the set (23) contains n^2 sums for $\delta=1$, n^2 sums for $\varepsilon=1$, and so forth, we have proved

THEOREM VIII. *The sums*

$$(24)\quad I_{\alpha\beta\gamma},\ I_{\alpha\beta 0\delta\varepsilon\cdots},\ I_{\alpha 01\delta\varepsilon\cdots},\ I_{\alpha\beta 2\delta\varepsilon\cdots}\qquad (\gamma=0,1,2,3;\ \delta+\varepsilon+\cdots=1)$$

form a complete and independent system for any number k of combined determinants of any order n.

IV

Dickson* has determined the types of general homogeneous polynomials which are expressible as determinants with linear elements. We shall now show that his main results follow very easily from our knowledge of the independence of sums of principal minors of determinants.

Let $f(x_1, x_2, \cdots, x_k)$ represent a general homogeneous polynomial of degree n in k variables. We assume that the coefficient of the nth power of one of the variables, say x_1, is unity. There is no loss of generality here since the coefficient can always be made unity by a linear transformation. If it is possible to express f as a determinant with linear elements, a combination of rows and columns will always bring that determinant to the form

$$(25)\qquad \begin{vmatrix} x_1+a_{11}x_2+b_{11}x_3+\cdots & a_{12}x_2+b_{12}x_3+\cdots & \cdots & a_{1n}x_2+b_{1n}x_3+\cdots \\ a_{21}x_2+b_{21}x_3+\cdots & x_1+a_{22}x_2+b_{22}x_3+\cdots & \cdots & a_{2n}x_2+b_{2n}x_3+\cdots \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}x_2+b_{n1}x_3+\cdots & a_{n2}x_2+b_{n2}x_3+\cdots & \cdots & x_1+a_{nn}x_2+b_{nn}x_3+\cdots \end{vmatrix}.$$

The expansion of (25) gives a polynomial in which each coefficient after the first is one of the sums $I_{\alpha\beta\gamma\cdots}$ and, conversely, each sum $I_{\alpha\beta\gamma\cdots}$ $(\alpha, \beta, \gamma, \cdots = 0, 1, 2, \cdots, n)$ appears as a coefficient, the number of subscripts in $I_{\alpha\beta\gamma\cdots}$ being $k-1$. Since this polynomial must be exactly f, we have a proof of

THEOREM IX. *A general polynomial of degree n in k variables cannot be expressed as a determinant with linear elements if it has in addition to x_1^n more than $(k-2)n^2+1$ terms if $k>3$ or more than $\frac{1}{2}(n^2+3n)$ terms if $k=3$.*

This is Dickson's first theorem except that he included the case $k=3$ in the general theorem. For $k=3$ the limit is $\frac{1}{2}(n^2+3n)$ which for $n>2$ is smaller than n^2+1, the limit for $k=3$ in the general theorem.

THEOREM X. *Any determinant of order n whose elements are linear homogeneous functions of $x_1, x_2, \cdots, x_k$ can be expressed in a canonical form involving not more than $(k-2)\,n^2+2$ parameters for $k>3$ and not more than $\frac{1}{2}(n^2+3n)+1$ parameters for $k=3$.*

This is Dickson's generalization of his first theorem except for the case $k=3$. Its proof follows immediately from the maximum number of independent sums which appear as the coefficients in the expansion of the determinant. The additional parameter arises from the fact that the

* These Transactions, vol. 22 (1921), pp. 167–179.

coefficient of x_1^n was assumed to be unity in the expansion of (25). If the expansion of the determinant does not contain the nth power of one of the variables, a linear transformation, independent of the coefficients of the form, will introduce such an nth power. The above proof then applies. The inverse of the linear transformation will restore the determinant to its original form without increasing the number of parameters.

Our knowledge concerning the particular sums $I_{\alpha\beta\gamma\cdots}$ which are independent gives immediately a limitation as to the forms which we may hope to express as determinants with linear elements. For $n = 2$ or 3, the coefficients $I_{\alpha\beta\gamma\cdots}$ are not all independent for $k > 4$. For $n > 3$ they are not all independent for $k > 3$. Thus, we have

THEOREM XI. *Unless $k = 2$ or 3, n any integer, or $k = 4$, $n = 2$, 3, the general form is not expressible in determinantal form.*

This is Dickson's second theorem.

Within the limitations just stated there is a one-to-one correspondence between the independent coefficients $I_{\alpha\beta\gamma\cdots}$ of the expanded form of (25) and the coefficients of the general form. Consequently, we have

THEOREM XII. *In the field of all complex numbers the general binary form, the general ternary form, the general quaternary quadratic form, and the general quaternary cubic form can be expressed in determinantal form.*

This is Dickson's fifth theorem, except that Dickson shows that the theorem is true for every binary form, every ternary form, and every quaternary quadratic form, regardless of the generality of the form.

The fact that the general quaternary cubic form can be expressed in determinantal form with linear elements leads at once to a theorem proved by H. Schröter* and also by Cremona;†

THEOREM XIII. *A sufficiently general cubic surface can be generated by three projective bundles of planes.*

For the determinantal form of the cubic equation says at once that there exist three parameters λ, μ, ν such that the three equations

$$\lambda l_{i1} + \mu l_{i2} + \nu l_{i3} = 0 \qquad (i = 1, 2, 3)$$

are satisfied by the coördinates of all points of the surface, where l_{ij} are expressions linear in x_1, x_2, x_3, x_4.

* Journal für Mathematik, vol. 62 (1863), p. 265.

† Ibid., vol. 68 (1868), p. 79.

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A NECESSARY AND SUFFICIENT CONDITION THAT TWO SURFACES BE APPLICABLE*

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It is well known that, in order that two surfaces be applicable, it is necessary that a map of the one upon the other exist so that geodesics correspond to geodesics and total curvature be preserved. It is also a familiar fact that neither of these conditions is alone sufficient. The primary purpose of this paper is to show that the two conditions taken together are sufficient, i. e., to prove the theorem

If two surfaces can be mapped geodesically so that total curvature is preserved, the surfaces are applicable.

1. The point of departure for the proof is Dini's theorem to the effect that, if two surfaces correspond by a geodesic map, then (a) each is mapped isometrically on the other or on a surface homothetic to the other, or (b) the two surfaces are surfaces of Liouville, whose linear elements can be put simultaneously into the forms

$$
(1)\qquad
\begin{aligned}
S_1:&\qquad ds_1^2 = (U+V)\,(du^2+dv^2),\\
S_2:&\qquad ds_2^2 = -\left(\frac{1}{U}+\frac{1}{V}\right)\left(\frac{du^2}{U}-\frac{dv^2}{V}\right),
\end{aligned}
$$

where U and V depend, respectively, on u and v alone, and corresponding points have the same curvilinear coördinates.

When we demand, further, that the geodesic map preserve total curvature, the surfaces in case (a) are obviously applicable. Case (b) is disposed of by the following lemma:

If two surfaces of Liouville with linear elements of the forms (1) *have the same total curvature in corresponding points, they are surfaces of constant curvature.*

For, it follows then that the surfaces are applicable, though not, it is to be noted, by the correspondence established by equations (1).

To prove the lemma, we compute the total curvatures K_1 and K_2 of S_1 and S_2 by means of the Gauss formula

* Presented to the Society, December 27, 1922.

$$
(2)\qquad
\begin{aligned}
K_1 &= \frac{1}{2(U+V)^3}[U'^2+V'^2-(U+V)(U''+V'')],\\
K_2 &= \frac{1}{4(U+V)^3}[-2(U^2V^2+UV^3)U''+(3UV^2+V^3)U'^2\\
&\qquad +2(U^2V^2+U^3V)V''-(3U^2V+U^3)V'^2].
\end{aligned}
$$

Setting $K_1 = K_2$, we have

$$
(3)\qquad
\begin{aligned}
&2(U+V-U^2V^2-UV^3)U''+(3UV^2+V^3-2)U'^2\\
&+2(U+V+U^2V^2+U^3V)V''-(3U^2V+U^3+2)V'^2 = 0.
\end{aligned}
$$

By means of the substitutions

$$
(4)\qquad x = U, \qquad y = V, \qquad X = U'^2, \qquad Y = V'^2,
$$

(3) becomes

$$
(5)\qquad
\begin{aligned}
&(x+y-x^2y^2-xy^3)\frac{dX}{dx}+(3xy^2+y^3-2)X\\
&+(x+y+x^2y^2+x^3y)\frac{dY}{dy}-(3x^2y+x^3+2)Y = 0,
\end{aligned}
$$

where X and Y depend, respectively, on x and y alone.

Differentiating (5) four times with respect to x, we get

$$
(x+y-x^2y^2-xy^3)\frac{d^5X}{dx^5}+(2-3y^3-5xy^2)\frac{d^4X}{dx^4} = 0.
$$

Since x and y are independent variables, it follows that $d^4X/dx^4 = 0$. Similarly, $d^4Y/dy^4 = 0$. Hence

$$
X = a_0+a_1x+a_2x^2+a_3x^3, \qquad Y = b_0+b_1y+b_2y^2+b_3y^3.
$$

Substituting these values of X, Y in (5) and equating collected coefficients of $x^m y^n$ to zero in the result, we get

$$
a_3 = b_3, \quad a_2 = -b_2, \quad a_1 = b_1, \quad a_0 = -b_0, \quad a_0 = -a_3.
$$

Hence, by virtue of (4),

$$
(6)\qquad
\begin{aligned}
U'^2 &= a_1U+a_2U^2+a_3(U^3-1),\\
V'^2 &= a_1V-a_2V^2+a_3(V^3+1).
\end{aligned}
$$

On substitution of these values in (2), we find that $K_1 = -\frac{1}{4}a_3$, so that K_1 and K_2 are constant, and our proof is complete.

2. Equations (6) form a special case of the equations

$$
\begin{aligned}
U'^2 &= a_0 + a_1 U + a_2 U^2 + a_3 U^3,\\
V'^2 &= -a_0 + a_1 V - a_2 V^2 + a_3 V^3.
\end{aligned}
\tag{7}
$$

For these more general values of U and V,

$$K_1 = -\frac{a_3}{4}, \qquad K_2 = \frac{a_0}{4}.$$

Conversely, if the curvature of either of the surfaces S_1, S_2 is constant, U and V must be defined by equations of the form (7). In proving this, there is no loss of generality in assuming S_1 to be the surface of constant curvature, for the relationship between S_1 and S_2 is reciprocal. Accordingly, we set $K_1 = -\frac{1}{4}a_3$ in (2), obtaining the equation

$$(U+V)(U''+V'') - (U'^2+V'^2) - \frac{a_3}{2}(U+V)^3 = 0.$$

On application of the substitutions (4), this reduces to

$$(x+y)\frac{dX}{dx} - 2X + (x+y)\frac{dY}{dy} - 2Y = a_3(x+y)^3. \tag{8}$$

Differentiating twice with respect to x, we get

$$\frac{d^3X}{dx^3} = 6a_3, \text{ whence } X = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Similarly,

$$\frac{d^3Y}{dy^3} = 6a_3, \quad \text{and} \quad Y = b_0 + b_1 y + b_2 y^2 + a_3 y^3.$$

Determining the coefficients in X, Y by substituting in (8), we come out with equations for U and V of the desired form (7). Incidentally we have also proved that the only surfaces which can be mapped geodesically on a surface of constant curvature are surfaces of constant curvature—Beltrami's theorem in a generalised form.

If in (1) U and V are replaced by $U+h$ and $V-h$, where h is an arbitrary constant, S_1 is unchanged, but S_2 is replaced by a one-parameter family of surfaces. The same substitutions in (7) leave a_3 unchanged and replace a_0 by $a_0+a_1 h+a_2 h^2+a_3 h^3$. Consequently, we have

$$K_1 = -\frac{a_3}{4}, \qquad K_2 = \frac{a_0+a_1 h+a_2 h^2+a_3 h^3}{4}.$$

Thus corresponding to a given surface S_1, that is, for a given set of values for the a's, there exist three surfaces S_2, in general distinct, of the same constant curvature as S_1, namely those corresponding to the three roots of the equation

$$a_3 h^3+a_2 h^2+a_1 h+a_0+a_3 = 0.$$

An exception arises in case S_1 is a developable ($a_3 = 0$); there then exist among the surfaces S_2 at most two developables.

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EXTENSIONS OF RELATIVE TENSORS*

BY

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1. **Introduction.** This paper is intended as an addendum to our previous paper on the *Geometry of paths*.‡ It contains the deduction of the formulas for the covariant derivatives and the higher extensions of relative tensors. These formulas are calculated by a process entirely analogous to that by which the corresponding formulas for ordinary tensors are obtained. Although the formulas are rather obvious the only one which we have observed in the literature is that for the covariant derivative of a relative tensor of the second order which was used by Einstein.§

2. **Relative tensors.** Let us denote by

$$x^i = f^i(\bar{x}^1, \cdots, \bar{x}^n) \tag{2.1}$$

an arbitrary analytic transformation of the coördinates $(x^1, \cdots, x^n)$ which can be written in the inverse form

$$\bar{x}^i = g^i(x^1, \cdots, x^n). \tag{2.2}$$

Since the inverse transformation (2.2) exists, the jacobian of the transformation (2.1)

$$\Delta = \left| \frac{\partial x}{\partial \bar{x}} \right| \tag{2.3}$$

does not vanish identically. As we shall later require the derivative $\partial \Delta / \partial \bar{x}^i$ of the jacobian Δ we here note that it is given by

$$\frac{\partial \Delta}{\partial \bar{x}^i} = \Delta \frac{\partial^2 x^\alpha}{\partial \bar{x}^i \partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^\alpha}. \tag{2.4}$$

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† National Research Fellow in Mathematical Physics, University of Chicago.

‡ These TRANSACTIONS, vol. 25 (1923), p. 551.

§ A. Einstein, *Zur allgemeinen Relativitätstheorie*, Sitzungsberichte der Preußischen Akademie der Wissenschaften, 1923, p. 32.

A set of functions $T^{lm\cdots n}_{ij\cdots k}$ will be said to form a relative tensor of weight K if it transforms according to the equations

$$(2.5)\qquad \overline{T}^{lm\cdots n}_{ij\cdots k} = \Delta^K T^{\mu\nu\cdots\delta}_{\alpha\beta\cdots\gamma} \frac{\partial \overline{x}^l}{\partial x^\mu} \frac{\partial \overline{x}^m}{\partial x^\nu} \cdots \frac{\partial \overline{x}^n}{\partial x^\delta} \frac{\partial x^\alpha}{\partial \overline{x}^i} \frac{\partial x^\beta}{\partial \overline{x}^j} \cdots \frac{\partial x^\gamma}{\partial \overline{x}^k}$$

when the variables are transformed by the equations (2.1). If $K = 0$, $T^{lm\cdots n}_{ij\cdots k}$ is an ordinary tensor. In case $K = 1$, $T^{lm\cdots n}_{ij\cdots k}$ is said to be a tensor density and is then usually written $\mathfrak{T}^{lm\cdots n}_{ij\cdots k}$. The justification of the name tensor density lies in the fact that the law of transformation of the integral

$$\int\int\cdots\int \mathfrak{T}^{lm\cdots n}_{ij\cdots k}\, d\, V$$

extended over a definite n-dimensional region approaches more and more closely the law of transformation of an ordinary tensor $T^{lm\cdots n}_{ij\cdots k}$ as the region of integration closes down on a point. Thus the tensor density $\mathfrak{T}^{lm\cdots n}_{ij\cdots k}$ represents a sort of tensor $T^{lm\cdots n}_{ij\cdots k}$ of weight zero per unit of coördinate volume $d\, V$.

We easily see that the algebraic processes of addition, multiplication, and contraction of ordinary tensors will also hold for the case of relative tensors.

3. **Covariant differentiation or first extension.** Let the relative tensor $T^{lm\cdots n}_{ij\cdots k}$ of weight K which is referred to the coördinates $(x^1, \cdots, x^n)$ be denoted by $t^{lm\cdots n}_{ij\cdots k}$ when referred to the system of normal coördinates $(y^1, \cdots, y^n)$ which are determined by the coördinates $(x^1, \cdots, x^n)$ and a point $(q^1, \cdots, q^n)$. We shall show that

$$(3.1)\qquad T^{lm\cdots n}_{ij\cdots k,p} = \left(\frac{\partial\, t^{lm\cdots n}_{ij\cdots k}}{\partial y^p}\right)_0,$$

where the derivative is evaluated at the origin of normal coördinates, defines a set of functions $T^{lm\cdots n}_{ij\cdots k,p}$ of $(x^1, \cdots, x^n)$ which are the components of a relative tensor of weight K. The relative tensor $T^{lm\cdots n}_{ij\cdots k,p}$ will be called the covariant derivative or first extension of the relative tensor $T^{lm\cdots n}_{ij\cdots k}$.

We denote by $(\overline{y}^1, \cdots, \overline{y}^n)$ the system of normal coördinates determined by the coördinates $(\overline{x}^1, \cdots, \overline{x}^n)$ and the point $(q^1, \cdots, q^n)$. Then

$$(3.2)\qquad \overline{y}^i = a^i_\alpha\, y^\alpha$$

where the a's are constants. Furthermore denoting by $\bar{t}^{lm\cdots n}_{ij\cdots k}$ the relative tensor $T^{lm\cdots n}_{ij\cdots k}$ when referred to the normal coördinates $(\bar{y}^1, \cdots, \bar{y}^n)$ we have

$$(3.3)\qquad \bar{t}^{lm\cdots n}_{ij\cdots k} = \Delta^K t^{\mu\nu\cdots\delta}_{\alpha\beta\cdots\gamma} \frac{\partial \bar{y}^l}{\partial y^\mu}\frac{\partial \bar{y}^m}{\partial y^\nu}\cdots\frac{\partial \bar{y}^n}{\partial y^\delta}\frac{\partial y^\alpha}{\partial \bar{y}^i}\frac{\partial y^\beta}{\partial \bar{y}^j}\cdots\frac{\partial y^\gamma}{\partial \bar{y}^k}.$$

In view of (3.2) the derivatives in (3.3) are constants and hence

$$(3.4)\qquad \frac{\partial \bar{t}^{lm\cdots n}_{ij\cdots k}}{\partial \bar{y}^p} = \Delta^K \frac{\partial t^{\mu\nu\cdots\delta}_{\alpha\beta\cdots\gamma}}{\partial y^\sigma} \frac{\partial \bar{y}^l}{\partial y^\mu}\frac{\partial \bar{y}^m}{\partial y^\nu}\cdots\frac{\partial \bar{y}^n}{\partial y^\delta}\frac{\partial y^\alpha}{\partial \bar{y}^i}\frac{\partial y^\beta}{\partial \bar{y}^j}\cdots\frac{\partial y^\gamma}{\partial \bar{y}^k}\frac{\partial y^\sigma}{\partial \bar{y}^p}.$$

Evaluating at the origin of normal coördinates we obtain

$$(3.5)\qquad \bar{T}^{lm\cdots n}_{ij\cdots k,p} = \Delta^K T^{\mu\nu\cdots\delta}_{\alpha\beta\cdots\gamma,\sigma} \frac{\partial \bar{x}^l}{\partial x^\mu}\frac{\partial \bar{x}^m}{\partial x^\nu}\cdots\frac{\partial \bar{x}^n}{\partial x^\delta}\frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial x^\beta}{\partial \bar{x}^j}\cdots\frac{\partial x^\gamma}{\partial \bar{x}^k}\frac{\partial x^\sigma}{\partial \bar{x}^p},$$

which shows that the set of functions $T^{lm\cdots n}_{ij\cdots k,p}$ constitute a relative tensor of weight K contravariant in the indices $(l, m, \cdots, n)$ and covariant in the indices $(i, j, \cdots, k, p)$.

We note that formulas analogous to those for the sum and product of two ordinary tensors likewise apply for the case of relative tensors.

To obtain the explicit formula involving the Γ's and their derivatives for the relative tensor $T^{lm\cdots n}_{ij\cdots k,p}$ we make use of the equations

$$(3.6)\qquad t^{lm\cdots n}_{ij\cdots k} = \Delta^K T^{\mu\nu\cdots\delta}_{\alpha\beta\cdots\gamma} \frac{\partial y^l}{\partial x^\mu}\frac{\partial y^m}{\partial x^\nu}\cdots\frac{\partial y^n}{\partial x^\delta}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\cdots\frac{\partial x^\gamma}{\partial y^k}.$$

Differentiating these equations and evaluating at the origin of normal coördinates we obtain

$$(3.7)\qquad \begin{aligned} T^{lm\cdots n}_{ij\cdots k,p} = \frac{\partial T^{lm\cdots n}_{ij\cdots k}}{\partial x^p} &+ T^{\alpha m\cdots n}_{ij\cdots k}\Gamma^l_{\alpha p} + \cdots + T^{lm\cdots\alpha}_{ij\cdots k}\Gamma^n_{\alpha p} \\ &- T^{lm\cdots n}_{\alpha j\cdots k}\Gamma^\alpha_{ip} - \cdots - T^{lm\cdots n}_{ij\cdots\alpha}\Gamma^\alpha_{kp} - K\,T^{lm\cdots n}_{ij\cdots k}\Gamma^\alpha_{\alpha p}. \end{aligned}$$

This is the general formula of covariant differentiation. By using the tensors $D^{\sigma ab\cdots n}_{\alpha\tau\beta\cdots\nu}$ and $E^{\alpha\tau\beta\cdots\nu}_{\sigma ab\cdots n}$ (cf. *The geometry of paths*, loc. cit.) the form (3.7) may be abbreviated. Thus

$$(3.8)\qquad \begin{aligned} T^{lm\cdots n}_{ij\cdots k,p} = \frac{\partial T^{lm\cdots n}_{ij\cdots k}}{\partial x^p} &+ \Gamma^\tau_{\sigma p} T^{\alpha\beta\cdots\gamma}_{ij\cdots k} D^{\sigma lm\cdots n}_{\alpha\tau\beta\cdots\gamma} \\ &- \Gamma^\sigma_{\tau p} T^{lm\cdots n}_{\alpha\beta\cdots\gamma} E^{\alpha\tau\beta\cdots\gamma}_{\sigma ij\cdots k} - K\,T^{lm\cdots n}_{ij\cdots k}\Gamma^\alpha_{\alpha p}. \end{aligned}$$

4. Higher extensions. By a process similar to that employed for the covariant derivative we may show that

$$(4.1) \qquad T^{lm\cdots n}_{ij\cdots k,p\cdots q} = \left(\frac{\partial^r t^{lm\cdots n}_{ij\cdots k}}{\partial y^p \cdots \partial y^q}\right)_0$$

defines a set of functions $T^{lm\cdots n}_{ij\cdots k,p\cdots q}$ of $(x^1,\cdots,x^n)$ which constitutes a relative tensor of weight K. The relative tensor $T^{lm\cdots n}_{ij\cdots k,p\cdots q}$ will be spoken of as the rth extension of the relative tensor $T^{lm\cdots n}_{ij\cdots k}$ provided that there are r indices $(p,\cdots,q)$. In case $r=1$ the extension reverts to the covariant derivative which we have already considered.

From its definition by means of the equations (4.1) we see that the extension $T^{lm\cdots n}_{ij\cdots k,p\cdots q}$ is symmetric in the indices $(p,\cdots,q)$. Thus

$$(4.2) \qquad T^{lm\cdots n}_{ij\cdots k,p\cdots q} = T^{lm\cdots n}_{ij\cdots k,u\cdots v}$$

where $(u,\cdots,v)$ denotes any permutation of the indices $(p,\cdots,q)$.

The formulas for the extension of the sum and product of two relative tensors are similar to the corresponding formulas for the extension of the sum and product of two ordinary tensors.

General formulas of extension $(r>1)$ may be calculated by the same process as that employed in the calculation of the general formula of covariant differentiation (3.7). The formula for the rth extension $T^{lm\cdots n}_{ij\cdots k,pqr\cdots uv}$ of a relative tensor $T^{lm\cdots n}_{ij\cdots k}$ of weight $K \neq 0$ involves the formulas for the first r extensions of $T^{lm\cdots n}_{ij\cdots k}$ considered as a tensor of weight zero. For we have

$$(4.3) \qquad t^{lm\cdots n}_{ij\cdots k} = \Delta^K f^{lm\cdots n}_{ij\cdots k},$$

where

$$f^{lm\cdots n}_{ij\cdots k} = T^{\mu\nu\cdots\delta}_{\alpha\beta\cdots\gamma}\frac{\partial y^l}{\partial x^\mu}\frac{\partial y^m}{\partial x^\nu}\cdots\frac{\partial y^n}{\partial x^\delta}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\cdots\frac{\partial x^\gamma}{\partial y^k}.$$

Differentiating (4.3) and evaluating at the origin of normal coördinates

$$(4.4) \qquad \begin{aligned} T^{lm\cdots n}_{ij\cdots k,pqr\cdots uv} = T^{lm\cdots n}_{ij\cdots k/pqr\cdots uv} &+ S(\Delta^K_p T^{lm\cdots n}_{ij\cdots k/qr\cdots uv}) \\ &+ S(\Delta^K_{pq} T^{lm\cdots n}_{ij\cdots k/r\cdots uv}) + \cdots \\ &+ S(\Delta^K_{pqr\cdots u} T^{lm\cdots n}_{ij\cdots k/v}) + \Delta^K_{pqr\cdots uv} T^{lm\cdots n}_{ij\cdots k}, \end{aligned}$$

where

$$T^{lm\cdots n}_{ij\cdots k/p\cdots q} = \left(\frac{\partial^r f^{lm\cdots n}_{ij\cdots k}}{\partial y^p \cdots \partial y^q}\right)_0,$$

$$\Delta^K_{p\cdots q} = \left(\frac{\partial^r \Delta^K}{\partial y^p \cdots \partial y^q}\right)_0,$$

and $S(\quad)$ denotes the sum of the different terms obtainable from the one in parenthesis by forming arbitrary combinations of the subscripts $(p, q, r, \cdots, u, v)$. The expressions $T^{lm\cdots n}_{ij\cdots k/p\cdots q}$ are given by the ordinary formulas for the rth extension of a tensor $T^{lm\cdots n}_{ij\cdots k}$ of weight zero, where r is the number of indices $(p, \cdots, q)$, but these expressions do not in general constitute a tensor. The quantities $\Delta^K_{p\cdots q}$ have the values

$$\begin{aligned}
\Delta^K_p &= -K\Gamma^\alpha_{\alpha p},\\
\Delta^K_{pq} &= -K\Gamma^\alpha_{\alpha pq} - K\Gamma^\alpha_{\beta p}\Gamma^\beta_{\alpha q} + K^2\Gamma^\alpha_{\alpha p}\Gamma^\beta_{\beta q},\\
\Delta^K_{pqr} &= -K\Gamma^\alpha_{\alpha pqr} - KS(\Gamma^\alpha_{\beta p}\Gamma^\beta_{\alpha qr}) - KS(\Gamma^\alpha_{\beta p}\Gamma^\beta_{\gamma q}\Gamma^\gamma_{\alpha r})\\
&\qquad + K^2S(\Gamma^\alpha_{\alpha p}\Gamma^\beta_{\beta qr}) + K^2S(\Gamma^\alpha_{\alpha p}\Gamma^\gamma_{\beta q}\Gamma^\beta_{\gamma r}) - K^3\Gamma^\alpha_{\alpha p}\Gamma^\beta_{\beta q}\Gamma^\gamma_{\gamma r},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}$$

in which Γ^i_{jkl} etc. are definite functions of the Γ^i_{jk} and their derivatives (cf. *The geometry of paths,* loc. cit., p. 561). Thus we may write

$$T^{lm\cdots n}_{ij\cdots k,\,p} = T^{lm\cdots n}_{ij\cdots k/p} - KT^{lm\cdots n}_{ij\cdots k}\Gamma^\alpha_{\alpha p} \tag{4.5}$$

in place of the formula (3.7).

Any formula of extension of a relative tensor $T^{lm\cdots n}_{ij\cdots k}$ of weight K may be obtained by substituting the proper values of $T^{lm\cdots n}_{ij\cdots k/p\cdots q}$ and $\Delta^K_{p\cdots q}$ in (4.4). A few particular expressions $T^{lm\cdots n}_{ij\cdots k/p\cdots q}$ are given in the *Geometry of paths* as formulas of extension of tensors of weight zero and these when substituted into the equations (4.4) together with the above values of $\Delta^K_{p\cdots q}$ give complete formulas in terms of the Γ^i_{jk} and their derivatives for the extensions of certain relative tensors of weight K.

GEOMETRIES OF PATHS FOR WHICH THE EQUATIONS OF THE PATHS ADMIT A QUADRATIC FIRST INTEGRAL*

BY

LUTHER PFAHLER EISENHART

1. A geometry of paths for a general space of n dimensions is based upon the conception that the paths are fundamental entities of the space. They are the integral curves of a system of differential equations

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{1.1}$$

where $x^i (i = 1, \cdots, n)$ are the coördinates of a point, $\Gamma^i_{jk} (= \Gamma^i_{kj})$ are functions of the x's and s is a parameter peculiar to each path. An important class of these geometries, which admits the Riemannian geometry as a sub-class, is that for which the equations (1.1) admit a quadratic first integral

$$g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \text{const.}, \tag{1.2}$$

where $g_{ij} (= g_{ji})$ are the components of a covariant tensor. Veblen and Thomas† have considered the problem, given a set of Γ's to determine whether equations (1.1) admit an integral (1.2); they have shown that its solution involves only algebraic processes. In this paper the converse problem is solved, namely to determine the Γ's so that (1.1) shall admit a given first integral (1.2); also the more general problem when the first integral is of the form

$$e^{\int \varphi_\alpha dx^\alpha} g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \text{const.},$$

where φ_α is a vector and the integral is taken along the path.

2. **General formulas.** If we put $x^i = \varphi^i (x'^1, \cdots, x'^n)$, thus introducing a new set of coördinates, equations (1.1) become

* Presented to the Society, March 1, 1924.

† These Transactions, vol. 25 (1923), pp. 599-608.

$$\frac{d^2 x'^i}{d s^2} + \Gamma'^i_{\alpha\beta} \frac{d x'^\alpha}{ds} \frac{d x'^\beta}{ds} = 0, \tag{2.1}$$

where

$$\frac{\partial^2 x^k}{\partial x'^\alpha \partial x'^\beta} + \Gamma^k_{ij} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} = \Gamma'^\sigma_{\alpha\beta} \frac{\partial x^k}{\partial x'^\sigma}. \tag{2.2}$$

Suppose now that g_{ij} are the components of a symmetric covariant tensor of the second order, that is

$$g'_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta}, \tag{2.3}$$

then by means of (2.2) it can be shown that the functions $g_{ij,k}$, defined by

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{i\alpha} \Gamma^\alpha_{jk} - g_{\alpha j} \Gamma^\alpha_{ik}, \tag{2.4}$$

are the components of a covariant tensor of the third order. As thus defined $g_{ij,k}$ is a generalization of the first covariant derivative of g_{ij}, the ordinary covariant derivative for a Riemann space being given by (2.4) when Γ^α_{jk} is replaced by $\left\{ \begin{matrix} \alpha \\ jk \end{matrix} \right\}$, the Christoffel symbol of the second kind formed with respect to the fundamental form of the space,* and similarly for Γ^α_{ik}.

The components g^{ij} of the contravariant tensor associate to g_{ij} are given by

$$g^{i\alpha} g_{j\alpha} = \delta^i_j, \tag{2.5}$$

where

$$\delta^i_j = 0 \text{ or } 1, \text{ as } i \neq j \text{ or } i = j. \tag{2.6}$$

As thus defined, g^{ij} is the cofactor of g_{ij} in the determinant $g = |g_{ij}|$ divided by g.

If we put

$$[ij, k] = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \tag{2.7}$$

and

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = g^{k\alpha} [ij, \alpha], \tag{2.8}$$

* Cf. Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 21.

then $[ij, k]$ and $\left\{ {k \atop ij} \right\}$ are the Christoffel symbols of the first and second kinds respectively formed with respect to the g's.

If we write also

$$\Gamma_{jk,i} = g_{\alpha i} \Gamma^{\alpha}_{jk}, \qquad \Gamma^{\alpha}_{jk} = g^{\alpha i} \Gamma_{jk,i}, \tag{2.9}$$

then from (2.4) and similar equations we have

$$g_{ki,j} + g_{jk,i} - g_{ij,k} = 2([ij, k] - \Gamma_{ij,k}). \tag{2.10}$$

From (2.3) we have by differentiation and suitable operations*

$$\frac{\partial^2 x^k}{\partial x'^{\alpha} \partial x'^{\beta}} + \left\{ {k \atop ij} \right\} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} = \left\{ {\sigma \atop \alpha\beta} \right\}' \frac{\partial x^k}{\partial x'^{\sigma}}, \tag{2.11}$$

where $\left\{ {\sigma \atop \alpha\beta} \right\}'$ is formed with respect to the g''s. Subtracting this equation from (2.2), we obtain

$$\Gamma'^{\sigma}_{\alpha\beta} - \left\{ {\sigma \atop \alpha\beta} \right\}' = \left(\Gamma^k_{ij} - \left\{ {k \atop ij} \right\} \right) \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} \frac{\partial x'^{\sigma}}{\partial x^k}.$$

Hence if we put

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + c_{ij}{}^k, \tag{2.12}$$

the functions $c_{ij}{}^k$ are the components of a tensor contravariant of the first order and covariant (and symmetric) of the second order. In consequence of (2.8) and (2.9) we have from (2.12)

$$\Gamma_{ij,k} = [ij, k] + c_{ijk}. \tag{2.13}$$

Hence (2.10) becomes

$$g_{ki,j} + g_{jk,i} - g_{ij,k} = -2 c_{ijk}. \tag{2.14}$$

If we add to this equation the two equations obtained from it by permuting i, j, k cyclically, we obtain

$$g_{ij,k} + g_{jk,i} + g_{ki,j} = -2(c_{ijk} + c_{jki} + c_{kij}). \tag{2.15}$$

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* Cf. Bianchi, *Lezioni*, vol. 1, p. 64.

3. **Quadratic first integrals.** When we express the condition that (1.2) be a first integral of (1.1), we obtain

$$g_{ij,k}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

Since this must be satisfied identically, we must have

$$g_{ij,k} + g_{jk,i} + g_{ki,j} = 0 \qquad (i, j, k = 1, \cdots, n). \tag{3.1}$$

The consistency of these equations is the necessary and sufficient condition that (1.2) be a first integral.*

We consider also the case when (1.1) admits a first integral of the form

$$e^{\int \varphi_\alpha dx^\alpha} g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \text{const.}, \tag{3.2}$$

where φ_α are the components of a vector and the integral $\int \varphi_\alpha\, dx^\alpha$ is taken along the path in question. Proceeding as above, we obtain the equations

$$g_{ij,k} + g_{jk,i} + g_{ki,j} + g_{ij}\,\varphi_k + g_{jk}\,\varphi_i + g_{ki}\,\varphi_j = 0. \tag{3.3}$$

Conversely, if these equations are consistent and yield a tensor g_{ij} and a vector φ_α, then (3.2) is a first integral of the corresponding equations (1.1).

It can be shown† that if we put

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} - \tfrac{1}{4}(\delta^k_i\,\varphi_j + \delta^k_j\,\varphi_i) \tag{3.4}$$

and define a parameter $\bar{s}$ for each path by means of the equation

$$\frac{d\bar{s}}{ds} = e^{-\frac{1}{2}\int \varphi_\alpha dx^\alpha}, \tag{3.5}$$

* Veblen and Thomas, loc. cit., pp. 599–608, give a complete treatment of the question of the consistency of equations (3.1).

† Annals of Mathematics, ser. 2, vol. 24 (1923), p. 376.

the equations

$$(3.6) \qquad \frac{d^2 x^k}{d\bar{s}^2} + \overline{\Gamma}^k_{ij} \frac{dx^i}{d\bar{s}} \frac{dx^j}{d\bar{s}} = 0$$

define the same curves as (1.1).

If we denote by $g_{ij,\bar{k}}$ the covariant derivative of g_{ij} given by the equation obtained by replacing Γ^k_{ij} by $\overline{\Gamma}^k_{ij}$ in (2.4), we have

$$(3.7) \qquad g_{ij,\bar{k}} = g_{ij,k} + \tfrac{1}{4}(2 g_{ij} \varphi_k + g_{jk} \varphi_i + g_{ki} \varphi_j).$$

Now (3.3) becomes

$$(3.8) \qquad g_{ij,\bar{k}} + g_{jk,\bar{i}} + g_{ki,\bar{j}} = 0,$$

and (3.2) reduces to the form (1.2) by means of (3.5).

The change (3.4) in the Γ's means a change in the affine connection of the space, but not in the paths themselves. Hence we have the theorem

When the equations of the paths of a space admit a first integral of the form (3.2), *by a change in the affine connection but not in the paths themselves the equations of the paths can be given a form which admits a first integral of the form* (1.2).

When, in particular,

$$(3.9) \qquad g_{ij,k} = -g_{ij} \varphi_k,$$

equations (3.3) are satisfied. This is Weyl's geometry, for it follows from (2.10) that

$$(3.10) \qquad \Gamma_{ij,k} = [i\,j,\,k] + \tfrac{1}{2}(g_{jk} \varphi_i + g_{ki} \varphi_j - g_{ij} \varphi_k).^*$$

Consequently, if the paths are taken as fundamental rather than the affine connection, it follows from (3.4) that if in (1.1) we take

$$(3.11) \qquad \Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + \tfrac{1}{4}(\delta^k_j \varphi_i + \delta^k_i \varphi_j - g_{ij} \varphi^k)$$

we have a geometry of the space with the same paths, whose equations admit the first integral (1.2).

Returning to the consideration of (3.1), we observe that from this equation and (2.15) we have the theorem

* Cf. *Space, Time and Matter*, p. 125.

A necessary and sufficient condition that equations (1.1) *admit a quadratic first integral* (1.2) *is that the tensor* c_{ijk} *defined by* (2.13) *satisfy the conditions*

$$c_{ijk} + c_{jki} + c_{kij} = 0. \tag{3.12}$$

4. **Determination of geometries of paths with quadratic first integrals.** Suppose that we have any symmetric tensor g_{ij} and a covariant tensor of the third order, a_{ijk}, symmetric in i and j. If we define a tensor, c_{ijk}, by means of the equations

$$c_{ijk} = 2a_{ijk} - a_{ikj} - a_{jki} \qquad (i, j, k = 1, \cdots, n), \tag{4.1}$$

it is by definition symmetric in i and j, and satisfies (3.12). If we define a set of Γ's by means of (2.13) in which $[ij, k]$ are formed with respect to the given tensor g_{ij} and c_{ijk} is given by (4.1), it follows from the theorem at the end of § 3 that the equations (1.1) admit the corresponding first integral (1.2).

Consider, conversely, the case when a geometry of this kind is given. Then the c_{ijk} as given by (2.13) satisfy (3.12). This condition is met, if we determine the components a_{ijk} of a tensor such that (4.1) hold for c_{ijk} known.

For each set of values of i, j, k all different, there are two equations (4.1), which are equivalent to

$$\begin{aligned} a_{ijk} - a_{kij} &= d_{jik}, \\ a_{jki} - a_{kij} &= d_{jki} \qquad (i, j, k = 1, \cdots, n;\ i, j, k \neq), \end{aligned} \tag{4.2}$$

where

$$d_{jik} = \tfrac{1}{3}(2c_{jik} + c_{jki}) = \tfrac{1}{3}(c_{jik} - c_{kij}), \tag{4.3}$$

the second and third expressions being equivalent because of (3.12). From (4.3) follow the identities

$$\begin{aligned} d_{jik} + d_{kij} &= 0, \\ d_{jik} + d_{kji} + d_{ikj} &= 0 \qquad (i, j, k = 1, \cdots, n;\ i, j, k \neq). \end{aligned} \tag{4.4}$$

In the general case, that is when c_{ijk} do not satisfy any conditions other than (3.12), there are, in consequence of (4.4), $n(n-1)(n-2)/3$ independent equations of the type (4.2).

From (4.1) and (3.12) we have also

$$a_{iij} - a_{iji} = d_{iij} \qquad (i, j = 1, \cdots, n;\ i, j \neq), \tag{4.5}$$

where

$$d_{iij} = \tfrac{1}{2} c_{iij} = \tfrac{1}{3} (c_{iij} - c_{jii}). \tag{4.6}$$

In the general case there are $n(n-1)$ independent equations of this type.

From (3.1) we have $g_{ii,i} = 0$, so that from (2.10) we have $\Gamma_{iii} = [ii, i]$ and consequently $c_{iii} = 0$. In this case (4.1) vanishes identically and consequently the components a_{iii} are not determined. Hence there are $n(n-1)(n+1)/3$ independent equations for the determination of the $n^2(n+1)/2$ components of a_{ijk}. Consequently there are $n\ (n+1)(n+2)/6$ components arbitrary: they are one for each case where i, j, k are different; one where two are different; and all of the type a_{iii}. Hence we have the theorem

A tensor g_{ij} and a tensor a_{ijk}, symmetric in i and j, determine a geometry of paths whose equations admit the corresponding first integral (1.2); *conversely, if a geometry is given whose equations admit a first integral* (1.2), $n(n+1)(n+2)/6$ *of the components a_{ijk} are arbitrary and the others can be found directly.*

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A GENERAL MEAN-VALUE THEOREM*

BY

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In a paper published in 1906†, Professor G. D. Birkhoff treated the mean-value and remainder theorems belonging to polynomial interpolation, in which the linear differential operator $u^{(n)}$ played a particular rôle. It is natural to expect that a generalization of many of the ideas of that paper may apply to the general linear differential operator of order n, and the author is attempting such a program. This generalization throws fundamentally new light on the theory of trigonometric interpolation.

A very elegant paper by G. Pólya‡ has just appeared treating mean-value theorems for the general operator in a restricted interval. It is the special aim of the present paper to develop a general mean-value theorem, and to show how it can be specialized to obtain Pólya's results.

We consider a linear differential expression of order n,

$$Lu \equiv u^{(n)}(x) + l_1(x)u^{(n-1)}(x) + \cdots + l_n(x)u(x),$$

where $l_1(x)$, $l_2(x), \cdots, l_n(x)$ are continuous functions, and $u(x)$ is continuous with its first $(n-1)$ derivatives, the nth derivative being piecewise continuous. All functions concerned are real. It is the purpose of this paper to obtain a necessary and sufficient condition for the change of sign of Lu in an interval in which u vanishes $(n+1)$ times.

More generally, the $(n+1)$ conditions implied in the vanishing of u may be replaced by an equal number of conditions involving also the derivatives of u. Let $x_0, x_1, \cdots, x_n$ be points of the closed interval (a, b), which points need not be all distinct, and let $k_0, k_1, \cdots, k_n$ be zero or positive integers not greater than $n-1$. We take then as $n+1$ conditions on u the relations

$$(A) \qquad u^{(k_i)}(x_i) = 0 \qquad (i = 0, 1, 2, \cdots, n).$$

* Presented to the Society, May 3, 1924.

† G. D. Birkhoff, *General mean-value and remainder theorems*, these Transactions, vol. 7 (1906), pp. 107–136. See also Bulletin of the American Mathematical Society, vol. 28 (1922), p. 5.

‡ G. Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, these Transactions, vol. 24 (1922), pp. 312–324.

Here $u^{(k_i)}(x)$ denotes the k_ith derivative of u, and $u^{(0)}(x)$ is the same as $u(x)$. We assume that no two of these equations are identical.

Now let $u_1, u_2, \cdots, u_n$ be n linearly independent solutions of the homogeneous equation

$$(1) \qquad Lu = 0.$$

For definiteness take them as the principal solutions for the point a; that is, solutions satisfying the conditions

$$\begin{array}{llll}
u_1(a) = 0, & u_1'(a) = 0, & \cdots, \quad u_1^{(n-2)}(a) = 0, & u_1^{(n-1)}(a) = 1, \\
u_2(a) = 0, & u_2'(a) = 0, & \cdots, \quad u_2^{(n-2)}(a) = 1, & u_2^{(n-1)}(a) = 0, \\
\cdots & \cdots & \cdots & \cdots \\
u_n(a) = 1, & u_n'(a) = 0, & \cdots, \quad u_n^{(n-2)}(a) = 0, & u_n^{(n-1)}(a) = 0.
\end{array}$$

We consider also the non-homogeneous equation

$$(2) \qquad Lu = \varphi(x),$$

where $\varphi(x)$ is piece-wise continuous in (a, b). The general solution of (2) is now obtained by Cauchy's method. Determine a solution of (1) which together with its first $n-2$ derivatives vanishes at a point t of (a, b), while the $(n-1)$th derivative has the value unity at that point. Denote the function by $g(x, t)$. It satisfies the n conditions

$$(3) \quad g(t,t) = 0, \quad g'(t,t) = 0, \quad \cdots, \quad g^{(n-2)}(t,t) = 0, \quad g^{(n-1)}(t,t) = 1.$$

Here the differentiation is with respect to the first argument.

It is known that the general solution of (2) may be written in the form

$$u(x) = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n + \tfrac{1}{2}\int_a^b \pm g(x,t)\,\varphi(t)\,dt.$$

Here $c_1, c_2, \cdots, c_n$ are arbitrary constants, and the sign before $g(x, t)$ is to be taken positive if $t < x$ and negative if $t > x$.* Considered as a function of t, $g(x, t)$ satisfies the equation adjoint to (1),

$$M(v) \equiv (-1)^n \frac{d^n v}{dx^n} + (-1)^{(n-1)} \frac{d^{n-1}}{dx^{n-1}}(l_1 v) + \cdots - \frac{d}{dx}(l_{n-1} v) + l_n v = 0.$$

* See for example D. A. Westfall, Dissertation, p. 16.

If now we express the fact that $u(x)$ satisfies the $n+1$ conditions (A), we obtain $n+1$ equations

$$(4)\quad c_1 u_1^{(k_i)}(x_i) + c_2 u_2^{(k_i)}(x_i) + \cdots + c_n u_n^{(k_i)}(x_i) + \tfrac{1}{2}\int_a^b \pm g^{(k_i)}(x_i, t)\varphi(t)\,dt = 0$$

$$(i = 0, 1, 2, \cdots, n).$$

Eliminating $c_1, c_2, \cdots, c_n$ from these equations, there results an equation of the form

$$(5)\qquad \int_a^b \Delta(t)\varphi(t)\,dt = 0,$$

where $\Delta(t)$ is the determinant

$$\left| \pm \tfrac{1}{2} g^{(k_i)}(x_i, t)\, u_1^{(k_i)}(x_i)\, u_2^{(k_i)}(x_i) \cdots u_n^{(k_i)}(x_i) \right| \qquad (i = 0, 1, \cdots, n).$$

Denote the cofactors of the elements of the first column by $\Delta_0, \Delta_1, \cdots, \Delta_n$, so that $\Delta(t)$ takes the form

$$\Delta(t) = \sum_{i=0}^{n} \pm \tfrac{1}{2} g^{(k_i)}(x_i, t)\Delta_i.$$

It is evident that $\Delta(t)$ depends in no way on the choice of the linearly independent solutions $u_1, u_2, \cdots, u_n$, but merely on the position of the points $x_0, x_1, \cdots, x_n$.

Now let us suppose that $\Delta(t)$ is not identically zero. Then the function u satisfying the conditions (A) can not be a solution of (1) unless it is identically zero; for a necessary and sufficient condition that there exist a solution of (1) not identically zero and satisfying the conditions (A) is precisely that

$$\Delta_i = 0 \qquad (i = 0, 1, \cdots, n).$$

Then $\varphi(t)$ is not identically zero, and we have at once from (5) a sufficient condition that Lu change sign in the interval (a, b), namely that $\Delta(t)$ should be a function of one sign in that interval. We may in particular take a and b as the two points of the set $x_0, x_1, \cdots, x_n$ which are farthest apart, and thus be assured that the change of sign of Lu occurs between these two points.

The condition is also necessary. Suppose that any function u with the required degree of continuity which is not identically zero and which satis-

fies the conditions (A) is such that Lu changes sign between the extreme points. It is desired to show that $\Delta(t)$ is a function of one sign not identically zero.

We note first that the Δ_i are not all zero; for if they were, there would be a solution u of (1) satisfying the conditions (A). This is impossible since Lu must change sign by hypothesis. In order to prove that $\Delta(t)$ does not vanish identically we must investigate its structure more closely.

If we denote by $v_1, v_2, \cdots, v_n$ the solutions adjoint to $u_1, u_2, \cdots, u_n$, we may write $g(x, t)$ as follows:*

$$g(x, t) = \sum_{i=1}^{n} u_i(x)\, v_i(t).$$

$\Delta(t)$ then takes the form

$$\Delta(t) = \tfrac{1}{2}\Big[v_1(t) \sum_{i=0}^{n} \pm u_1^{(k_i)}(x_i)\,\Delta_i + v_2(t) \sum_{i=0}^{n} \pm u_2^{(k_i)}(x_i)\,\Delta_i + \cdots$$
$$\cdots + v_n(t) \sum_{i=0}^{n} \pm u_n^{(k_i)}(x_i)\,\Delta_i\Big].$$

If $\Delta(t)$ were identically zero, each summation in the above expression would be zero, since $v_1, v_2, \cdots, v_n$ are linearly independent. Now by taking t in all possible positions with respect to the points $x_0, x_1, \cdots, x_n$, various combinations of signs in each summation are obtained.

It would follow then that

$$u_j^{(k_i)}(x_i)\,\Delta_i = 0 \qquad (i = 0, 1, \cdots, n;\ j = 1, 2, \cdots, n).$$

Not all the Δ_i are zero. Suppose that $\Delta_m \neq 0$. Then it would follow that

$$u_j^{(k_m)}(x_m) = 0 \qquad (j = 1, 2, \cdots, n).$$

The Wronskian of $u_1, u_2, \cdots, u_n$ would vanish at the point x_m, contrary to the assumption that $u_1, u_2, \cdots, u_n$ are linearly independent. $\Delta(t)$ can not therefore vanish identically.

Suppose now that $\Delta(t)$ changes sign between the extreme points. It is then possible to choose a continuous† function of one sign $\bar{\varphi}(t)$ such that

$$\int_a^b \bar{\varphi}(t)\,\Delta(t)\,dt = 0. \tag{6}$$

* See for example, Darboux, *Théorie des Surfaces*, vol. 2, p. 106.

† Indeed $\bar{\varphi}$ may be a piece-wise continuous function made up of straight lines parallel to the x-axis.

Now consider the equations (4) in which $\varphi(t)$ is replaced by $\bar{\varphi}(t)$. From these $n+1$ equations, pick out that set of n equations which has Δ_m for its determinant. Since $\Delta_m \neq 0$, the set of equations has a unique solution in $c_1, c_2, \cdots, c_n$. If we substitute this solution in the equation

$$\bar{u}^{(k_m)}(x) = c_1 u_1(x) + \cdots + c_n u_n(x) + \tfrac{1}{2}\int_a^b \pm g^{(k_m)}(x, t)\,\bar{\varphi}(t)\,dt,$$

we obtain an equation of the form

$$\bar{u}^{(k_m)}(x) = \int_a^b G^{(k_m)}(x, t)\,\bar{\varphi}(t)\,dt,$$

where $G(x, t)$ may be identified with the Green's function corresponding to the boundary conditions (A).*

It is seen that

$$G^{(k_m)}(x_m, t) = \Delta(t).$$

It follows from (6) that

$$\bar{u}^{(k_m)}(x_m) = 0.$$

By its very definition $\bar{u}(x)$ is seen to satisfy the remainder of the conditions (A). $L\bar{u}$ must therefore change sign. But $L\bar{u}$ is $\bar{\varphi}$, a function which does not change sign. We have thus completed the proof of the following

THEOREM I. *A necessary and sufficient condition that Lu change sign in an interval in which u (having the required degree of continuity† and not identically zero) satisfies the conditions (A) is that $\Delta(t)$ be a function of one sign not identically zero in that interval.*

G. Pólya obtains certain theorems concerning the vanishing of Lu. We may obtain these results from Theorem I.

With Pólya we say that the property W holds for the operator Lu in an open interval (a, b) if there exist solutions of (1), $h_1, h_2, \cdots, h_{n-1}$, such that the following functions do not vanish in (a, b):

$$W_1 = h_1, \qquad W_2 = W(h_1, h_2) = \begin{vmatrix} h_1 & h_2 \\ h_1' & h_2' \end{vmatrix}, \quad \cdots$$

$$\cdots, \qquad W_{n-1} = W(h_1, \cdots, h_{n-1}) = \begin{vmatrix} h_1 & h_2 & \cdots & h_{n-1} \\ h_1' & h_2' & \cdots & h_{n-1}' \\ \cdot & \cdot & \cdot & \cdot \\ h_1^{(n-2)} & h_2^{(n-2)} & \cdots & h_{n-1}^{(n-2)} \end{vmatrix}.$$

* C. E. Wilder, these Transactions, vol. 18 (1917), p. 416.

† The restrictions on the nth derivative of u might be made lighter as is done in Birkhoff's paper, loc. cit.

Pólya considers the special case of the conditions (A) which involves the vanishing of $u(x)$ at points of the interval that are distinct or coincident. Consider r points

$$x_1 < x_2 < \cdots < x_r, \qquad r \leqq n+1.$$

Suppose that $u(x)$ vanishes m_i times at a point x_i $(m_i \leqq n-1)$:

$$(B) \qquad \begin{array}{l} u(x_i) = u'(x_i) = \cdots = u^{(m_i-1)}(x_i) = 0, \quad u^{(m_i)}(x_i) \neq 0, \quad i = 1, 2, \cdots, r; \\ \sum_{i=1}^{r} m_i = n+1. \end{array}$$

If $x_1, x_2, \cdots, x_r$ lie in the interval in which the property W holds, $\Delta(t)$ is a function of one sign. To prove this we investigate the structure of $\Delta(t)$. In each of the $r-1$ intervals, $\Delta(t)$ is a solution of the adjoint equation, continuous with its first n derivatives. It is only at a point x_i that a discontinuity may occur. Such a discontinuity is caused by a change in the ambiguous sign before $g^{(k_i)}(x_i, t)$ as t passes over x_i. However, it is only when $g^{(k_i)}(x_i, x_i)$ is not zero that such a discontinuity is introduced. We can now show that at a point x_i where $u(x)$ vanishes m_i times $\Delta(t)$ is continuous with its first $n - m_i - 1$ derivatives.

It is known* that

$$\frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial t^\nu} g(x, t) \bigg|_{t=x} = \sum_{i=1}^{n} u_i^{(\mu)}(x) v_i^{(\nu)}(t) \bigg|_{t=x} = 0, \qquad \mu + \nu < n-1,$$
$$= (-1)^\nu, \quad \mu + \nu = n-1.$$

At x_i, μ may equal $m_i - 1$, so that

$$\frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial t^\nu} g(x_i, x_i) = 0, \qquad \mu < m_i, \qquad \nu < n - m_i.$$

It follows then that $\Delta(t)$ is continuous at x_i with its first $n - m_i - 1$ derivatives.

Now if $t < x_i$, $i = 1, 2, \cdots, r$, then all the ambiguous signs in $\Delta(t)$ are positive; if $t > x_i$, $i = 1, 2, \cdots, r$, they are all negative. In either case $\Delta(t)$ is identically zero for the interval considered, since

$$\sum_{i=0}^{n} u_j^{(k_i)}(x_i) \Delta_i = 0 \qquad (j = 1, 2, \cdots, n),$$

* See Schlesinger, *Lineare Differential-Gleichungen*, vol. 1, p. 63.

the expression on the left being a determinant with two columns equal. Now since $\Delta(t)$ is continuous with its first $n-m_1-1$ derivatives at x_1, it follows that $\Delta(t)$ has $n-m_1$ zeros at x_1. It has $n-m_r$ zeros at x_r.

Suppose first that

$$m_1 = m_2 = \cdots = m_{n+1} = 1, \qquad x_r = x_{n+1}.$$

Then $\Delta(t)$ is continuous throughout with its first $n-2$ derivatives and has $n-1$ zeros in each of the points x_1 and x_r. Suppose it were not a function of one sign. It would have in all at least $2n-1$ zeros in the closed interval (x_1, x_r).

Now recall that $M(v)$ can be "factored" as follows:

$$M(v) \equiv \frac{(-1)^n}{W_1}\frac{d}{dx}\frac{W_1^2}{W_2W_0}\frac{d}{dx}\frac{W_2^2}{W_3W_1}\frac{d}{dx}\cdots\frac{d}{dx}\frac{W_{n-1}^2}{W_nW_{n-2}}\frac{d}{dx}\frac{W_n}{W_{n-1}}v,$$

where $W_0 = 1$.* Each of the quantities $W_0, W_1, \cdots, W_n$ does not vanish in the interval (x_1, x_{n+1}), so that we may apply Rolle's theorem. If $\Delta(t)$ vanished $2n-1$ times in (x_1, x_{n+1}), the function

$$\psi(t) = \frac{W_1^2}{W_2W_0}\frac{d}{dx}\frac{W_2^2}{W_1W_3}\frac{d}{dx}\cdots\frac{d}{dx}\frac{W_n}{W_{n-1}}\Delta(t)$$

would change sign at least n times *inside* the interval. But $M(\Delta(t))$ is identically zero, so that $\psi(t)$ is constant in any interval in which it is continuous. Hence $\psi(t)$ can change sign only at the points $x_2, x_3, \cdots, x_n$, where it is discontinuous. There are only $n-1$ such points, so that it must be concluded that $\Delta(t)$ is a function of one sign in (x_1, x_{n+1}).

In order to obtain the proof in the general case we shall need the following

LEMMA. *If a function $f(x)$ is continuous in an interval in which $f'(x)$ is continuous except for l finite jumps, and if $f'(x)$ can have at most N zeros in the interval, $f(x)$ can have at most $N+l+1$ zeros there.*

Proof. If $f(x)$ had $N+l+2$ zeros, $f'(x)$ would have $N+l+1$ zeros and discontinuous changes of sign. At most l of these can be discontinuous changes of sign, and $f'(x)$ would have $N+1$ zeros contrary to hypothesis.

Now denote by s_i the number of integers $m_2, m_3, \cdots, m_{r-1}$, which are equal to i. Then

$$s_1 + s_2 + \cdots = r-2. \tag{7}$$

* See Schlesinger, loc. cit., p. 58.

Define a function $\bar{\Delta}^{(k)}(t)$ by the equation

$$\bar{\Delta}^{(k)}(t) = \frac{W_{n-k}^2}{W_{n-k-1}W_{n-k+1}} \frac{d}{dx} \frac{W_{n-k+1}^2}{W_{n-k}W_{n-k+2}} \frac{d}{dx} \cdots \frac{d}{dx} \frac{W_n}{W_{n-1}} \Delta(t).$$

Now apply the lemma to $\bar{\Delta}^{(n-2)}$ in each of the $r-s_1-1$ intervals in which it is continuous. We may evidently treat all these intervals simultaneously and take l equal to the total number of discontinuities of $\bar{\Delta}^{(n-1)}$, viz., $r-2$. N must be zero since $\bar{\Delta}^{(n-1)}$ is constant where it is continuous.* We conclude that $\bar{\Delta}^{(n-2)}$ can vanish at most $r-1$ times.

Now apply the lemma to $\bar{\Delta}^{(n-3)}$. Here $N=r-1$ and $l=r-s_1-2$. Then $\bar{\Delta}^{(n-3)}$ can have at most $2(r-1)-s_1$ zeros. $\bar{\Delta}^{(n-4)}$ can have at most $3(r-1)-2s_1-s_2$. Proceeding in this way we see that $\Delta(t)$ has at most

$$(n-1)(r-1)-(n-2)s_1-(n-3)s_2-\cdots$$

zeros. Now $\Delta(t)$ has

$$n-m_1+n-m_r = n-1+s_1+2s_2+3s_3+\cdots$$

zeros at the end points x_1 and x_r, and this number is precisely equal to the maximum number of zeros $\Delta(t)$ can have, since by virtue of (7)

$$(n-1)(r-1)-(n-2)s_1-(n-3)s_2-\cdots = n-1+s_1+2s_2+\cdots.$$

The proof is thus complete that $\Delta(t)$ can not change sign.

Before proceeding to the converse of this theorem, let us draw several further inferences.

THEOREM II. *In an interval in which the property W holds, the coefficient of* $g^{(m_i-1)}(x_i, t)$ *in* $\Delta(t)$ *can not vanish,* $i=1,2,\cdots,r$.

For if it did $\Delta(t)$ would be continuous with its first $n-m_i$ derivatives at x_i, and by means of the lemma a contradiction would be reached as before.

COROLLARY. *No solution of equation* (1) *can vanish n times in an interval in which the property W holds unless it is identically zero.*

For the vanishing of the coefficient Δ_i of Theorem II is precisely the condition that there exist a solution of (1) not identically zero passing through the n points (some of which may be coincident) involved in Δ_i.

* $\bar{\Delta}^{(n-1)}(t)$ is not identically zero in any interval between x_1 and x_r.

These n points were arbitrary in the interval so that the corollary is established.

Now let us show that if $\Delta(t)$ is a function of one sign in an interval $a \leqq x < b$ for every set of conditions (B) in this interval, then the property W holds in the interval $a < x < b$. We prove first that no solution of (1) can vanish n times in the interval $a \leqq x < b$. For suppose there were such a solution u. Let x_r be the point of vanishing nearest b, and let x' be a point between x_r and b. Now determine a solution w of the differential system

$$Lw = 1,$$
$$w(x') = w'(x') = \cdots = w^{(n-1)}(x') = 0.$$

Form the function $\bar{u}(x)$ which is

$$\begin{array}{ll} u(x) + Mw(x), & x' \leqq x < b, \\ u(x), & a \leqq x < x', \end{array}$$

where M is a constant to be determined. We wish to show that M can be so determined that $\bar{u}(x)$ vanishes $n+1$ times in $a \leqq x < b$. Now $u(x') \neq 0$ since $x_r < x'$, and hence it follows that $\bar{u}(x')$ has the sign of $u(x')$. Choose a point x'' between x' and b for which $w(x'')$ is not zero. Such a point exists since $w(x) \not\equiv 0$ in any interval. Now choose M so that $\bar{u}(x'')$ will have a sign opposite to that of $\bar{u}(x')$; $\bar{u}$ will then vanish between x' and x''. $\bar{u}$ has the required degree of continuity to apply the mean-value theorem, and $\Delta(t)$ is a function of one sign. Hence $L\bar{u}$ must change sign. But $L\bar{u}$ is equal to zero in the interval $a < x < x'$ and to M in the interval $x' < x < b$, and does not change sign. The contradiction shows that u can not vanish n times in $a \leqq x < b$.

But if no solution of (1) vanishes n times in $a \leqq x < b$, it is a simple matter to show that the property W holds in $a < x < b$. For the principal solutions for the point a, $u_1, u_2, \cdots, u_n$, are suitable functions. The Wronskian $W_k = W(u_1, u_2, \cdots, u_k)$ does not vanish in $a < x < b$. For suppose it vanished at a point c of that interval. Then a function

$$c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$$

could be determined not identically zero and having k zeros in the point c. But this function would have $n - k$ zeros in a, and a total of n zeros in $a \leqq x < b$. This is impossible. We may now state the following

THEOREM III. *A necessary and sufficient condition that the vanishing of a function u (with the required degree of continuity and not identically zero) at $n+1$ arbitrary points of an interval (a, b) should imply the change of sign of Lu at an intermediate point is that the property W hold in (a, b).*

A simple example will suffice to show that Theorem I is stronger than Theorem III. Take

$$Lu = u'' + u,$$

$$x_0 = 0 < x_1 < x_2.$$

Then

$$\Delta(t) = \frac{1}{2} \begin{vmatrix} \sin t & 0 & 1 \\ \pm \sin(x_1 - t) & \sin x_1 & \cos x_1 \\ \sin(x_2 - t) & \sin x_2 & \cos x_2 \end{vmatrix},$$

$$\Delta(t) = \sin t \sin(x_1 - x_2), \qquad 0 < t < x_1,$$

$$= \sin x_1 \sin(t - x_2), \qquad x_1 < t < x_2.$$

Suppose now that $x_1 < \pi$ and $x_2 - x_1 < \pi$.

Then

$$\Delta(t) < 0, \qquad 0 < t < x_2.$$

$\Delta(t)$ is a function of one sign in the interval $(0, x_2)$ which may clearly be of length greater than π. (In fact it may be as near to 2π as we like.) Yet the property W can not hold in any interval of length greater than π, in as much as some solution of (1) will vanish twice in such an interval. This example suggests possible generalizations of Pólya's results.

It should be pointed out that Theorem I might easily be made to apply to the most general linear boundary conditions.

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